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Graph colorings and realization of manifolds as leaves

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Graph colorings and realization of manifolds as leaves

D. Jesús Antonio Álvarez López

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Não me esqueci de nada, mãe.
Guardo a tua voz dentro de mim.
E deixo-te as rosas.

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Si conobbero. Lui conobbe lei e se stesso, perché in verità non s'era mai saputo. E lei conobbe lui e se stessa, perché pur essendosi saputa sempre, mai s'era potuta riconoscere così.

Italo Calvino

Canta, rula:
alto, quedo.
Na nenez
do meu pai.

Rodolfo Alonso

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Chapter 1

Introduction

This thesis has two main parts. The first one is devoted to show that, for any infinite connected (repetitive) graph X with finite maximum vertex degree $\deg X < \infty$, there exists a (repetitive) limit-aperiodic coloring by at most $\deg X$ colors. Several direct consequences of this theorem are also derived, like the existence of (repetitive) limit-aperiodic colorings of any (repetitive) tiling of a Riemannian manifold. The second part is devoted to prove that any (repetitive) Riemannian manifold of bounded geometry can be isometrically realized as leaf of a compact Riemannian (minimal) foliated space, whose leaves have no holonomy. This also uses the previous result about colorings, but it also requires much more technical work concerning the space of pointed Riemannian manifolds with the topology defined by the C^∞ convergence. The following sections contain more precise descriptions.

1.1 Graph colorings

The results of this section will be in the publication [9].

Let (X, E) (or simply X) be a connected simple[1] (undirected) graph. Given a set F of natural numbers, a coloring $\phi\colon X \to F$ is said to be *non-periodic*, *aperiodic*[2] or *distinguishing* if there is no nontrivial automorphism of X preserving ϕ. The *distinguishing number*, denoted by $D(X)$, is the smallest positive integer such that there is some non-periodic coloring ϕ of X by $D(X)$ colors. This concept was introduced in [2] by Albertson and Collins, and the calculation of $D(X)$ (or bounds thereof) for many families of graphs has been the subject of much research in recent years (see e.g. [47], [26]).

Another connected simple graph Y is said to be a *limit* of X if, using the graph distance d_Y, for every $n \in \mathbb{N}$ and $y \in Y$, we can find an isomorphic copy of the ball

[1] Recall that a graph is *simple* if there is at most one edge joining every pair of vertices.

[2] In some publications, this term was used with the meaning of what is called limit-aperiodicity in this thesis.

$B_Y(y,n)$ of Y inside X. Analogously, we can define when a colored graph (Y,ψ) is the *limit* of (X,ϕ). A coloring $\phi\colon X\to F$ is *limit-aperiodic* or *limit-distinguishing* if every limit colored graph (Y,ψ) is distinguishing, and the *limit-distinguishing number*, denoted by $D_L(X)$, is the least $n\in\mathbb{N}$ such that there is a limit-distinguishing coloring by n colors.

A graph X (respectively, a colored graph (X,ϕ)) is *repetitive* if every finite pattern of X (respectively, of (X,ϕ)) appears uniformly in X with respect to the graph distance d_X. Let X be a graph with maximum degree $\deg X<\infty$. The first main result of this thesis states that $D_L(X)\le\deg X$ for every infinite graph as above.

If X is a finite graph, then limit-aperiodicity is equivalent to aperiodicity, and we have $D_L(X)=D(X)$. In this case, it was proved in [48] that $D(X)\le\deg X$ except in the following cases, where $D(X)\le\deg X+1$: the complete graph K_n on n vertices ($n\ge 2$), the (n,n)-bipartite graph $K_{n,n}$ ($n\ge 1$), and the cyclic graph C_5 with 5 vertices. So we will restrict our attention to infinite graphs.

Theorem 1.1.1. *Let X be an infinite connected simple graph with $\Delta:=\deg X<\infty$. Then there is a limit-aperiodic coloring ϕ of X by Δ colors. Moreover, if X is repetitive, then there is a repetitive limit-aperiodic coloring by Δ colors.*

Let X be a connected simple graph. A coloring of its edge set, $\phi:E(X)\to F$, is called an (*edge-*) *coloring* of X, and (X,ϕ) is called an *edge-colored graph*. The concepts of (*pointed*) *isomorphisms*, *isomorphic* (pointed) edge-colored graphs, and *automorphism* groups of (pointed) edge-colored graphs are obvious generalizations of those corresponding to vertex-colored (pointed) graphs. Then we can define the notion of *aperiodic*, *limit-aperiodic* and *repetitive* edge-colored graph in the same way as we did for vertex-colored graphs in Section 2.3.

Suppose now that X is infinite and has finite maximum degree $\deg X<\infty$. Then the associated *line graph*, denoted by $L(X)$, is defined as follows. Every vertex in $L(X)$ represents an edge in X, and two vertices in $L(X)$ are adjacent if and only if the corresponding edges in X share a vertex. $L(X)$ is a simple graph satisfying $\deg L(X)\le 2(\deg X-1)$. By the definition of $L(X)$, there is an obvious correspondence between vertex colorings of $L(X)$ and edge colorings of X. By the Whitney strong isomorphism theorem [76], we have that X and $L(X)$ univocally determine each other up to isomorphism, and moreover there is a one-to-one correspondence between the isomorphisms of X and those of $L(X)$. The construction of $L(X)$ from X is local in nature, in the sense that, given an edge e of X with corresponding vertex $v_e\in L(X)$, the pointed isomorphism class of $(B_{L(X)}(v_e,r),v_e)$ is determined by the pointed isomorphism class of $(B_X(u,r+1),u)$, where u is any vertex adjacent to e. This fact and the Whitney strong isomorphism theorem imply that, if $L(Y)$ is a limit of $L(X)$, then the graph Y is a limit of X. Then we can derive the following result.

Theorem 1.1.2. *Let X be a simple infinite (repetitive) graph of finite maximum degree $\deg X < \infty$. Then X admits a (repetitive) limit-aperiodic edge coloring by $\deg L(X) \leq 2(\deg X - 1)$ colors.*

The following informal discussion illustrates how this theorem can be proved. Let ϕ be a limit-aperiodic coloring of $L(X)$. Then there is an induced edge coloring $\hat{\phi}$ on X. Let $(Y, \hat{\psi})$ be an edge-colored graph that is a limit of $(X, \hat{\phi})$, and suppose that $(Y, \hat{\psi})$ has some non-trivial symmetry $h \neq \mathrm{id}$. Then, by the preceding discussion, $\hat{\psi}$ induces a vertex-coloring $(L(Y), \psi)$, and h induces a non trivial symmetry $\hat{h}$ of $(L(Y), \psi)$. But $(L(Y), \psi)$ has to be a limit of $(L(X), \phi)$, contradicting the assumption that ϕ is limit-aperiodic. The repetitivity of the coloring can be deduced along similar lines.

The analogue of the distinguishing number when one considers edge colorings instead of vertex colorings is called the *distinguishing index*, denoted by $D'(X)$. The value of $D'(X)$ for some families of graphs was studied in [3], [4]. In [15], the bound $D'(X) \leq \deg X$ was proved for X infinite. We can define the *limit-distinguishing index* of a connected simple graph X, denoted by $D'_L(X)$, as the least number of colors needed to produce a limit-aperiodic edge coloring of X. Then Theorem 1.1.2 provides a bound for the limit-distinguishing index.

In fact, we think that a proof similar to that presented on Chapter 3 may produce a limit-aperiodic edge coloring by $\deg X$ colors for any simple infinite graph X with $\deg X < \infty$. Choose a point $p \in X$. We can construct an aperiodic edge coloring using $\deg X$ colors using the same ideas of Proposition 3.6.27. Namely, we use the color 0 to distinguish the point p and use the remaining colors using the obvious analogue of BFS-orderings for edges (see Section 3.6.4). Then, following the lines of Chapter 3, one may be able to prove the following.

Conjecture 1.1.3. Let X be a simple infinite graph of bounded maximum degree $\deg X < \infty$. Then there is a limit-aperiodic edge coloring by $\deg X$ colors. Moreover, if X is repetitive, then the coloring can be taken to be repetitive.

An interesting application of Theorem 1.1.1 is the existence of limit-aperiodic colored tilings. For the sake of simplicity, consider a tiling T of an n-dimensional (connected) Riemannian manifold with corners, M, by tiles meeting face to face, taken from (i.e., isometric to) a finite set of prototiles, $\mathcal{T}$, consisting of compact Riemannian manifolds of dimension n with boundary (see [14] for the definition of tilings of more general spaces). The tiling isomorphisms of (M, T) are the isometries of M that map tiles to tiles. Using such tiling isomorphisms, there are obvious versions of limit-aperiodicity and repetitivity in this setting. Also, colored tilings and face-colored tilings have an obvious meaning, as well as their limit-aperiodicity and repetitivity.

We can associate to T a graph X, with one vertex v_t for each tile $t \in T$, and declaring that v_t is adjacent to $v_{t'}$ if and only if t and t' meet at some $(n-1)$-dimensional face. Then X is an infinite graph of bounded degree, and every tiling isomorphism of T (a tiling preserving isometry of (M,T)) induces an isomorphism of X. Let $\phi\colon X \to [\deg X] := \{0,1,\dots,\deg X - 1\}$ be a limit-aperiodic coloring of X. This coloring induces a limit-aperiodic colored tiling T' by colored tiles taken from the finite set of colored prototiles $\mathcal{T}' := \mathcal{T} \times [\deg X]$. Moreover, if the tiling is repetitive, then the resulting colored tiling can be chosen to be repetitive as well. In summation, we have the following result.

Theorem 1.1.4. *Let T be a tiling by finitely many prototiles meeting face to face, and let Δ be the maximum number of $(n-1)$-dimensional faces of the prototiles. Then there is a limit-aperiodic coloring of the tiling by Δ colors. If T is repetitive, then the coloring can be assumed to be repetitive.*

Coloring the faces of the tiles instead of the tiles themselves, we can also derive the following result from Theorem 1.1.2.

Theorem 1.1.5. *Let T be a tiling by finitely many prototiles meeting face to face, and let Δ be the maximum number of $(n-1)$-dimensional faces of the prototiles. Then there is a limit-aperiodic edge coloring of the tiling by $2(\Delta-1)$ colors. If T is repetitive, then the edge coloring can be assumed to be repetitive.*

Since the coloring of the faces of a tile is a local matching rule, it can be enforced by shape. Then Theorem 1.1.5 has the following consequence. If a (repetitive) Riemannian manifold M admits a tiling T as before, with a finite set of prototiles $\mathcal{T}$, then there is another finite set of prototiles $\mathcal{T}'$ such that there is a limit-aperiodic (repetitive) tiling T' of M using prototiles from $\mathcal{T}'$.

Theorem 1.1.1 will also be used to realize Riemannian manifolds of bounded geometry as leaves of compact Riemannian foliated spaces. This will be explained in detail in Section 1.2.

To get an intuitive idea of the proof of Theorem 1.1.1, let us consider the case of breaking non-trivial symmetries on finite graphs. Suppose then that X is finite, with maximum degree $\deg X$. There is an easy way to construct an aperiodic coloring that goes as follows. Pick any point $x \in X$, and assign the color 0 to it. Now, if x is the only point with color 0, then any graph automorphism $h\colon X \to X$ must fix x. The sphere $S(x,1)$ has at most $\deg X - 1$ points, so if we color the sphere so that no two different points share their color, then h must fix all points in $S(x,1)$. This procedure can be continued by induction using an order relation on X, and in the end we obtain an aperiodic coloring by $\deg X$ colors. Moreover, if X has a pair of points x, y that are far enough, it is easy to see that we can reuse the color 0 to obtain many different aperiodic colorings from this construction.

With the previous construction in mind, roughly speaking, our proof proceeds as follows. First, we divide our graph $X \equiv X_{-1}$ into finite connected clusters of bounded size, such that their centers form a Delone set $X_0 \subset X_{-1}$. Then, for each cluster, we construct a large enough amount of different colorings $\phi^i_{-1,x}$. If we choose a coloring $\phi^i_{-1,x}$ for each $x \in X_0$, we can see this as defining a coloring in X_0 sending x to i. The set X_0 can be endowed with a graph structure so that, if the induced coloring in X_0 sends points that are close to different colors, we obtain the following partial result for the combination of the colorings $\phi^i_{-1,x}$ and some $R, S > 0$: if there is a color-preserving isomorphism between $(B_X(x,R), x)$ and $(B_X(y,R), y)$, then either $x = y$ or $d(x,y) > S$.

The limit-aperiodicity condition is precisely a countable family of conditions of this type. Thus, we generalize the preceding discussion to divide X_0 into clusters, defining a graph $X_1 \subset X_0$ such that colorings in the clusters define a coloring in X_1, and so on. Using a diagonal argument, we obtain the desired coloring.

1.2 Realizability of manifolds as leaves

The results of this section will be also in the publication [9].

Recall that a foliated space $X \equiv (X, \mathcal{F})$ of dimension n is a topological space X equipped with a partition $\mathcal{F}$ into connected manifolds (leaves) so that X can be locally described as a product $B \times Z$, where B is an open ball in $\mathbb{R}^n$ and Z any topological space (local transversal), and the slices $B \times \{*\}$ correspond to open sets in the leaves. This $\mathcal{F}$ is called a foliated structure or lamination. Foliated spaces are usually assumed to be Polish[3] to get better properties. Many basic notions about foliations can be obviously extended to foliated spaces, like foliated charts, plaques, foliated atlas, holonomy pseudogroup, holonomy group and holonomy covering of the leaves, minimality, transitivity, foliated maps, etc. Some basic results can be extended as well; for instance, there is an obvious version of the Reeb local stability theorem, and the union of leaves without holonomy is a meager subset if X is second countable. Interesting classes of foliated spaces show up in several areas of mathematics, like in dynamics, arithmetics, tessellations, graphs and foliation theory (minimal sets).

A C^∞ foliated structure is given by a foliated atlas whose changes of coordinates are leafwise C^∞, with ambient-space-continuous leafwise derivatives of arbitrary order. This gives rise to the concept of C^∞ foliated space. To emphasize the difference, the foliated structure underlying a C^∞ foliated structure may be called topological. On a

[3] Recall that a topological space is *Polish* if it separable and completely metrizable

C^∞ foliated space $X \equiv (X, \mathcal{F})$, the concept of C^∞ function is defined by requiring that its local expressions, using foliated coordinates, are leafwise C^∞, with ambient-space-continuous leafwise partial derivatives of arbitrary order. C^∞ bundles and sections also make sense on X, defined by requiring that their local descriptions are given by C^∞ functions in the above sense. For instance, the tangent bundle TX (or $T\mathcal{F}$) is the C^∞ vector bundle on X that consists of the vectors tangent to the leaves, and a Riemannian metric on X consists of Riemannian metrics on the leaves fitting together nicely to form a C^∞ section on X. This gives rise to the concept of Riemannian foliated space.

In particular, if X is a manifold, then $(X, \mathcal{F})$ is a foliated manifold, and $\mathcal{F}$ is called a foliation.

The second problem addressed in this thesis is the realization of Riemannian manifolds as leaves of compact Riemannian foliated spaces. This is a variation of the problem of realizing manifolds as leaves of compact foliated manifolds, which has a long history with celebrated contributions of great mathematicians.

Foliation theory, as a genuine area of research, was begun by Reeb and Ehreshman and Haefliger. Reeb constructed the first foliation of S^3 using what is now called its Reeb component. Later, Novikov proved that any foliation of S^3 must contain at least one Reeb component. These ideas, together with the structure of the topology of flows (Poincaré-Bendixon theory, the Denjoy flow and the Cherry flow of the torus) took a definite position in mathematical research in the early 1970's, with the contributions of more mathematicians like Hirsch, Thurston, Plante, Mossu, Pelletier, Anosov, Ruelle-Sullivan, Raymond, Ghys, Inaba, Duminy, etc. In particular, as one of the first natural questions, it was asked by Sullivan [74] and Sondow [72] which manifolds can be realized as leaves of compact manifolds. Answering this question, it was proved that any surface can be realized as a leaf of a codimension one foliation on a closed manifold [20], but this fails in higher dimension [33], [49], [11], [73], [70].

Any leaf of a smooth compact foliated manifold $(M, \mathcal{F})$ has an induced quasi-isometric class of Riemannian metrics, represented by the restriction of any Riemannian metric on M. Then, as a metric version of this problem, it was also natural to ask which quasi-isometry types can be realized as leaves of compact foliated manifolds. In fact this problem has an obvious extension to smooth compact foliated spaces. An interesting publication about such metric structure of the leaves was written by D. Cass [21], who gave the first published results relating the recurrence properties of leaves of foliations with their quasi-isometry types, and who quoted an unpublished result of Gromov, which was later developed in [6].

There are examples of connected Riemannian manifolds of bounded geometry whose quasi-isometry type cannot be realized as leaves of foliations of codimension one on closed manifolds [11], [78], [68], [69]. For a nice survey on the historical account of these developments, see [46].

Bounded geometry plays an important role in this development. Recall that Riemannian manifold M is said to be of bounded geometry when it has a positive injectivity radius, and the m-th covariant derivative of the curvature tensor has uniformly bounded norm for all order m; in particular, M is complete by the positivity of the injectivity radius. The following are typical examples where bounded geometry holds: coverings of closed connected Riemannian manifolds, connected Lie groups with left invariant metrics, and leaves of compact Riemannian foliated spaces. More examples can be produced by using compactly supported perturbations of given Riemannian manifolds of bounded geometry. In fact, any smooth manifold admits a metric of bounded geometry [35]. Contrasting with the indicated constructions of "non-leaves of bounded geometry" in codimension one, we show the following theorem, which is the second main result of this thesis.

Theorem 1.2.1. *Any connected (repetitive) Riemannian manifold of bounded geometry is isometric to a leaf of some compact (minimal) Riemannian foliated space without holonomy.*

In the above theorem, a Riemannian manifold is repetitive if it satisfies the obvious Riemannian analogue of the repetitivity condition for graphs. Thus the general study the leaves of (minimal) compact Riemannian foliated spaces without holonomy is the study of (repetitive) Riemannian manifolds of bounded geometry. Since every smooth manifold admits Riemannian metrics of bounded geometry [35], we get the following consequence.

Corollary 1.2.2. *Any smooth manifold M can be realized as a leaf of a compact foliated space X without holonomy.*

It is commonly accepted that Theorem 1.2.1 should be true, at least without assuming that the ambient space has no holonomy, and that it should follow by using the closure of the canonical embedding of the manifold into the Gromov space $\mathcal{M}_*$ of pointed proper metric spaces [36], [37, Chapter 3], or, better, into its smooth version, the space $\mathcal{M}_*^\infty(n)$ of isometry classes of pointed complete connected Riemannian n-manifolds with the topology defined by the C^∞ convergence ([60, Chapter 10, Section 3.2] and Theorem 1.3.2). However, to the author's knowledge, no complete proof had been given prior to the publication of [5] and [6].

1.3 Gromov space of pointed Riemannian manifolds

The results of this section were published in [6].

For any $n \in \mathbb{N}$ (we adopt the convention that $0 \in \mathbb{N}$), let $\mathcal{M}_*(n)$ denote the set of isometry classes, $[M, x]$, of pointed complete connected Riemannian n-manifolds, (M, x). [4] With this assumption, $\mathcal{M}_*(n)$ is a well defined set. This set is interesting only for $n \geq 2$ because $\mathcal{M}_*(0) = \{[\{0\}, 0]\}$ and $\mathcal{M}_*(1) = \{[\mathbb{R}, 0], [\mathbb{S}^1, 1]\}$. The set $\mathcal{M}_*(n)$ can be considered as a subset of the Gromov space $\mathcal{M}_*$ of isometry classes of pointed proper metric spaces [36], [37, Chapter 3]. However it is interesting to consider a finer topology on $\mathcal{M}_*(n)$, taking the differentiable structure into account. For that purpose, the following notion of C^∞ convergence was defined on $\mathcal{M}_*(n)$.

Definition 1.3.1 (See e.g. [60, Chapter 10, Section 3.2])**.** For each $m \in \mathbb{N}$, a sequence $[M_i, x_i] \in \mathcal{M}_*(n)$ is said to be C^m *convergent* to $[M, x] \in \mathcal{M}_*(n)$ if, for each compact domain $\Omega \subset M$ containing x, there are pointed C^{m+1} embeddings $\phi_i : (\Omega, x) \to (M_i, x_i)$ for large enough i such that $\phi_i^* g_i \to g|_\Omega$ as $i \to \infty$ with respect to the C^m topology [44, Chapter 2]. If $[M_i, x_i]$ is C^m convergent to $[M, x]$ for all m, then it is said that $[M_i, x_i]$ is C^∞ *convergent* to $[M, x]$.

Here, a *domain* in M is a connected C^∞ submanifold, possibly with boundary, of the same dimension as M.

It is admitted that C^∞ convergence defines a topology on $\mathcal{M}_*(n)$ [59]. However we are not aware of any proof in the literature, prior to the publication of [6], showing that it satisfies the conditions to describe a topology [54], [39] (see also [52] and [53] if C^∞ convergence were defined with nets or filters). This is only proved on subspaces defined by manifolds of equi-bounded geometry, where the C^∞ convergence coincides with convergence in $\mathcal{M}_*$ [55] (see also [60, Chapter 10]). The first main theorem of this section is the following.

Theorem 1.3.2. *The C^∞ convergence in $\mathcal{M}_*(n)$ describes a Polish topology.*

The topology given by Theorem 1.3.2 will be called the C^∞ *topology* on $\mathcal{M}_*(n)$, and the corresponding space is denoted by $\mathcal{M}_*^\infty(n)$. The closure operator induced by this topology is denoted by Cl^∞.

For each complete connected Riemannian n-manifold M, there is a canonical continuous map $\iota : M \to \mathcal{M}_*^\infty(n)$ given by $\iota(x) = [M, x]$, which induces a continuous injective map $\bar{\iota} : \mathrm{Iso}(M)\backslash M \to \mathcal{M}_*^\infty(n)$, where $\mathrm{Iso}(M)$ denotes the isometry group of M. The

[4] The cardinality of each complete connected Riemannian n-manifold is less than or equal to the cardinality of the continuum, and therefore it may be assumed that its underlying set is contained in $\mathbb{R}$.

more explicit notation ι_M and $\bar\iota_M$ may be also used. The images of the maps ι_M form a natural partition of $\mathcal{M}_*^\infty(n)$, denoted by $\mathcal{F}_*(n)$.

The proof of Theorem 1.2.1 was motivated by the following intuition: one could hope that, for a Riemannian manifold of bounded geometry M, the closure $\mathrm{Cl}^\infty(\iota(M))$ would be a compact Riemannian foliated space and $\iota(M)$ would be a leaf isometric to M. Unfortunately, this approach does not work for several reasons. First, $\iota(M)$ need not be isometric to M in general, since $\mathrm{Iso}(M)$ may not be trivial. Secondly, even if $\mathrm{Cl}^\infty(\iota(M))$ is indeed a compact Riemannian foliated space, there may be leaves with non-trivial holonomy. To avoid the former problem, we need to consider a modification of $\mathcal{M}_*(n)$ that avoids the singularities induced by existence of non-trivial isometries.

A Riemannian manifold, M, is said to be *non-periodic* if $\mathrm{Iso}(M) = \{\mathrm{id}_M\}$, and is said to be *locally non-periodic* if each point $x \in M$ has a neighborhood U_x such that

$$\{\, h \in \mathrm{Iso}(M) \mid h(x) \in U_x \,\} = \{\mathrm{id}_M\}\ .$$

Let $\mathcal{M}_{*,\mathrm{np}}(n)$ and $\mathcal{M}_{*,\mathrm{lnp}}(n)$ be the $\mathcal{F}_*(n)$-saturated subsets of $\mathcal{M}_*(n)$ defined by non-periodic and locally non-periodic manifolds, respectively. The notation $\mathcal{M}^\infty_{*,\mathrm{np}}(n)$ and $\mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$ is used when these sets are equipped with the restriction of the C^∞ topology. The restrictions of $\mathcal{F}_*(n)$ to $\mathcal{M}_{*,\mathrm{np}}(n)$ and $\mathcal{M}_{*,\mathrm{lnp}}(n)$ are respectively denoted by $\mathcal{F}_{*,\mathrm{np}}(n)$ and $\mathcal{F}_{*,\mathrm{lnp}}(n)$. Note that $\mathcal{M}_{*,\mathrm{np}}(0) = \{[\{0\},0]\}$ and $\mathcal{M}_{*,\mathrm{lnp}}(1) = \emptyset$.

On the other hand, let $\mathcal{M}^\infty_{*,\mathrm{c}}(n)$ (respectively, $\widehat{\mathcal{M}}^\infty_{*,\mathrm{o}}(n)$) be the $\mathcal{F}_*(n)$-saturated subspace of $\widehat{\mathcal{M}}_*(n)$ consisting of classes $[M,x]$ such that M is compact (respectively, open). Observe that, if $[N,y]$ is close enough to any $[M,x] \in \mathcal{M}^\infty_{*,\mathrm{c}}(n)$, then N is diffeomorphic to M. Thus $\mathcal{M}^\infty_{*,\mathrm{c}}(n)$ is open in $\mathcal{M}_*(n)$, and therefore $\mathcal{M}^\infty_{*,\mathrm{o}}(n)$ is closed. Hence these are Polish subspaces of $\mathcal{M}_*(n)$, as well as their intersections with any Polish subspace. The intersection of $\mathcal{M}^\infty_{*,\mathrm{c/o}}(n)$ and $\mathcal{M}^\infty_{*,\mathrm{(l)np}}(n)$ is denoted by $\mathcal{M}^\infty_{*,\mathrm{(l)np,c/o}}(n)$. The restrictions of $\mathcal{F}_*(n)$ to $\mathcal{M}_{*,\mathrm{c/o}}(n)$ and $\mathcal{M}_{*,\mathrm{(l)np,c/o}}(n)$ are denoted by $\mathcal{F}_{*,\mathrm{c/o}}(n)$ and $\mathcal{F}_{*,\mathrm{(l)np,c/o}}(n)$, respectively. The main theorem of this chapter is the following.

Theorem 1.3.3. *The following properties hold for $n \geq 2$:*

(i) $\mathcal{M}_{*,\mathrm{lnp}}(n)$ *is Polish and dense in* $\mathcal{M}_*^\infty(n)$.

(ii) $\mathcal{M}^\infty_{*,\mathrm{lnp}}(n) \equiv (\mathcal{M}^\infty_{*,\mathrm{lnp}}(n), \mathcal{F}_{*,\mathrm{lnp}}(n))$ *is a foliated space of dimension n.*

(iii) $\mathcal{F}_{*,\mathrm{lnp,o}}(n)$ *is transitive.*

(iv) *The foliated space $\mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$ has canonical C^∞ and Riemannian structures such that $\bar\iota : \mathrm{Iso}(M)\backslash M \to \iota(M)$ is an isometry for every locally non-periodic, complete, connected Riemannian manifold M.*

(v) *For any locally non-periodic complete connected Riemannian manifold M, the quotient map $M \to \mathrm{Iso}(M)\backslash M$ corresponds to the holonomy covering of the leaf $\iota(M)$ by $\bar\iota : \mathrm{Iso}(M)\backslash M \to \iota(M)$. In particular, the set $\mathcal{M}_{*,\mathrm{np}}(n)$ is the union of leaves of $\mathcal{M}_{*,\mathrm{lnp}}^\infty(n)$ with trivial holonomy groups.*

The following result states a universal property of $\mathcal{M}_{*,\mathrm{lnp}}^\infty(n)$, which involves certain property called covering-determination (Definition 5.10.1).

Theorem 1.3.4. *Let X be a sequential Riemannian foliated space of dimension $n \geq 2$ whose leaves are complete. Then X is isometric to a saturated subspace of $\mathcal{M}_{*,\mathrm{lnp}}^\infty(n)$ if and only if it is covering-determined.*

Recall that a space X is called *sequential* if a subset $A \subset X$ is open whenever each convergent sequence $x_n \to x \in A$ in X eventually belongs to A. For instance, first countable spaces are sequential. This condition could be removed by using convergence of nets or filters instead of sequences.

$\mathcal{M}_{*,\mathrm{lnp}}^\infty(n)$ is used to prove the following result about realizations of Riemannian manifolds as leaves. It involves the obvious Riemannian versions of the conditions of being aperiodic or repetitive, which are standard for tilings or graphs (see e.g. [32,38,62]), and a weak version of aperiodicity (Definitions 5.10.4 and 5.10.6).

Theorem 1.3.5. *The following properties hold for a complete connected Riemannian manifold M of bounded geometry and dimension $n \geq 2$:*

(i) *M is non-periodic and has a (repetitive) weakly aperiodic connected covering if and only if it is isometric to a dense leaf of a (minimal) covering-determined compact sequential Riemannian foliated space.*

(ii) *If M is non-periodic (and repetitive), then it is isometric to a dense leaf of a (minimal) covering-determined compact sequential Riemannian foliated space whose leaves have trivial holonomy groups.*

1.4 Desingularization of $\mathcal{M}_*^\infty(n)$

The results of this section are in [5] and [9].

Fix a separable Hilbert space $\mathbb{E}$ and any natural n. Consider pairs (M, f) and triples (M, f, x), where M is a complete connected Riemannian n-manifold, $f \in C^\infty(M, \mathbb{E})$ and $x \in M$. An *equivalence* $\phi : (M, f) \to (N, h)$ is an isometry $\phi : M \to N$ such that $\phi^* h = f$. If moreover distinguished points, $x \in M$ and $y \in N$, are preserved, then $\phi : (M, f, x) \to (N, h, y)$ is called a *pointed equivalence*. The group of self equivalences of

(M, f) is denoted by $\operatorname{Iso}(M, f)$. If there is a pointed equivalence $(M, f, x) \to (N, h, y)$, then the triples (M, f, x) and (N, h, y) are declared to be *equivalent*. The equivalence class of each (M, f, x) is denoted by $[M, f, x]$. Let $\widehat{\mathcal{M}}_*(n)$ denote the set[5] of such equivalence classes.

Definition 1.4.1. For each $m \in \mathbb{N}$, a sequence $[M_i, f_i, x_i]$ in $\widehat{\mathcal{M}}_*(n)$ is said to be C^m *convergent* to $[M, f, x] \in \widehat{\mathcal{M}}_*(n)$ if, for each compact domain $\Omega \subset M$ containing x, there is a pointed C^{m+1} embedding $\phi_i : (\Omega, x) \to (M_i, x_i)$ for each large enough i such that $\phi_i^* g_i \to g|_\Omega$ and $\phi_i^* f_i \to f|_\Omega$ as $i \to \infty$ with respect to the C^m topology [44, Chapter 2]. If $[M_i, f_i, x_i]$ is C^m convergent to $[M, f, x]$ for all m, then it is said that $[M_i, f_i, x_i]$ is C^∞ *convergent* to $[M, f, x]$.

It is not completely obvious that this C^∞ convergence satisfies the conditions to define a topology [54], [39]. Thus the following result is not trivial.

Theorem 1.4.2. *The C^∞ convergence in $\widehat{\mathcal{M}}_*(n)$ describes a Polish topology.*

The topology given by Theorem 1.4.2 will be called the C^∞ *topology*, and the corresponding space is denoted by $\widehat{\mathcal{M}}_*^\infty(n)$. The closure operator in this space will be denoted by $\widehat{\operatorname{Cl}}_\infty$. The following maps are canonical and continuous: a *forgetful* map $\widehat{\mathcal{M}}_*^\infty(n) \to \mathcal{M}_*^\infty(n)$, $[M, f, x] \mapsto [M, x]$, and an *evaluation* map $\operatorname{ev} : \widehat{\mathcal{M}}_*^\infty(n) \to \mathbb{E}$, $[M, f, x] \mapsto f(x)$. Note that $\operatorname{ev} : \widehat{\mathcal{M}}_*(0) \to \mathbb{E}$ is a homeomorphism. Moreover, for each complete connected Riemannian n-manifold M and any $f \in C^\infty(M, \mathbb{E})$, there is a canonical continuous map $\hat\iota_{M,f} : M \to \widehat{\mathcal{M}}_*^\infty(n)$, given by $\hat\iota_{M,f}(x) = [M, f, x]$, which induces a continuous injection $\bar\iota_{M,f} : \operatorname{Iso}(M, f)\backslash M \to \mathcal{M}_*^\infty(n)$. The images of the maps $\hat\iota_{M,f}$ form a natural partition of $\widehat{\mathcal{M}}_*^\infty(n)$, denoted by $\widehat{\mathcal{F}}_*(n)$. Let $C^\infty_{\mathrm{imm}}(M, \mathbb{E})$ be the set of C^∞ immersions $M \to \mathbb{E}$, and let $\widehat{\mathcal{M}}_{*,\mathrm{imm}}^\infty(n)$ be the $\widehat{\mathcal{F}}_*(n)$-saturated subspace of $\widehat{\mathcal{M}}_*(n)$ consisting of classes $[M, f, x]$ with $f \in C^\infty_{\mathrm{imm}}(M, \mathbb{E})$. The restriction of $\widehat{\mathcal{F}}_*(n)$ to $\widehat{\mathcal{M}}_{*,\mathrm{imm}}(n)$ is denoted by $\widehat{\mathcal{F}}_{*,\mathrm{imm}}(n)$. Observe that the canonical projection $M \to \operatorname{Iso}(M, f)\backslash M$ is a covering map if $f \in C^\infty_{\mathrm{imm}}(M, \mathbb{E})$.

Let $\widehat{\mathcal{M}}_{*,\mathrm{c}}^\infty(n)$ (respectively, $\widehat{\mathcal{M}}_{*,\mathrm{o}}^\infty(n)$) be the $\widehat{\mathcal{F}}_*(n)$-saturated subspace of $\widehat{\mathcal{M}}_*(n)$ consisting of classes $[M, f, x]$ such that M is compact (respectively, open). Observe that, if $[N, h, y]$ is close enough to any $[M, f, x] \in \widehat{\mathcal{M}}_{*,\mathrm{c}}^\infty(n)$, then N is diffeomorphic to M. Thus $\widehat{\mathcal{M}}_{*,\mathrm{c}}^\infty(n)$ is open in $\widehat{\mathcal{M}}_*(n)$, and therefore $\widehat{\mathcal{M}}_{*,\mathrm{o}}^\infty(n)$ is closed. Hence these are Polish subspaces of $\widehat{\mathcal{M}}_*(n)$, as well as their intersections with any Polish subspace. Let $\widehat{\mathcal{M}}_{*,\mathrm{imm,c/o}}^\infty(n) = \widehat{\mathcal{M}}_{*,\mathrm{c/o}}^\infty(n) \cap \widehat{\mathcal{M}}_{*,\mathrm{imm}}^\infty(n)$. The restrictions of $\widehat{\mathcal{F}}_*(n)$ to the subsets $\widehat{\mathcal{M}}_{*,\mathrm{c/o}}(n)$ and $\widehat{\mathcal{M}}_{*,\mathrm{imm,c/o}}(n)$ are denoted by $\widehat{\mathcal{F}}_{*,\mathrm{c/o}}(n)$ and $\widehat{\mathcal{F}}_{*,\mathrm{imm,c/o}}(n)$, respectively.

[5] Like in the cases of $\mathcal{M}_*$ and $\mathcal{M}_*^\infty(n)$, without loss of generality, it can be assumed that the underlying set of any such M is contained in $\mathbb{R}$, so that $\widehat{\mathcal{M}}_*(n)$ becomes a well defined set.

Theorem 1.4.3. *The following properties hold:*

(i) $\widehat{\mathcal{M}}_{*,\mathrm{imm}}^\infty(n)$ *is Polish and dense in* $\widehat{\mathcal{M}}_*^\infty(n)$.

(ii) $\widehat{\mathcal{F}}_{*,\mathrm{imm}}(n)$ *is a foliated structure of dimension* n.

(iii) $\widehat{\mathcal{F}}_{*,\mathrm{imm,o}}(n)$ *is transitive.*

(iv) *There is a unique* C^∞ *foliated structure* $\widehat{\mathcal{F}}_{*,\mathrm{imm}}^\infty(n)$ *on* $\widehat{\mathcal{M}}_{*,\mathrm{imm}}^\infty(n)$*, whose underlying topological foliated structure is* $\widehat{\mathcal{F}}_{*,\mathrm{imm}}(n)$*, such that* $\mathrm{ev} : \widehat{\mathcal{M}}_{*,\mathrm{imm}}^\infty(n) \to \mathbb{E}$ *is a* C^∞ *immersion.*

(v) *There is a unique Riemannian metric on* $\widehat{\mathcal{M}}_{*,\mathrm{imm}}^\infty(n) \equiv (\widehat{\mathcal{M}}_{*,\mathrm{imm}}^\infty(n), \widehat{\mathcal{F}}_{*,\mathrm{imm}}^\infty(n))$ *such that* $\iota_{M,f} : M \to \hat{\iota}_{M,f}$ *is a local isometry for all complete connected Riemannian* n*-manifold* M *and* $f \in C_{\mathrm{imm}}^\infty(M, \mathbb{E})$.

(vi) *For all* M *and* f *as above, the map* $\hat{\iota}_{M,f} : M \to \mathrm{im}\,\hat{\iota}_{M,f}$ *is the holonomy covering of the leaf* $\mathrm{im}\,\hat{\iota}_{M,f}$.

It is possible to give a version of Theorem 1.4.3 closer to Theorem 1.3.3, using the subspace $\widehat{\mathcal{M}}_{*,\mathrm{lnp}}^\infty(n)$ consisting of the classes $[M, f, x]$ such that $M \to \mathrm{Iso}(M, f)\backslash M$ is a covering map. Such a result could be proved with the obvious adaptation of the proof of Theorem 1.3.3, using the exponential map to define foliated charts. Instead, we have opted for studying $\widehat{\mathcal{M}}_{*,\mathrm{imm}}^\infty(n)$ because, in this case, the immersions f directly provide foliated charts.

The following result states that $\widehat{\mathcal{M}}_{*,\mathrm{imm}}^\infty(n)$ is universal among the class of Polish Riemannian foliated spaces that satisfy a condition called covering-continuity (Definition 6.4.1).

Theorem 1.4.4. *A Polish Riemannian foliated space* X *of dimension* n *with complete leaves is isometric to a saturated Riemannian foliated subspace of* $\widehat{\mathcal{M}}_{*,\mathrm{imm}}^\infty(n)$ *if and only if* X *is covering-continuous.*

In Theorem 1.4.4, when X consists of a single leaf M, the isometric injection of M into $\widehat{\mathcal{M}}_{*,\mathrm{imm}}^\infty(n)$ is $\hat{\iota}_{M,f}$ for any C^∞ embedding $f : M \to \mathbb{E}$. If moreover M is of bounded geometry, then f can be chosen according to the following result.

Proposition 1.4.5. *For any (repetitive) connected Riemannian manifold* M *of bounded geometry, there is some (repetitive) limit-aperiodic function* $f \in C^\infty(M, \mathbb{E})$ *of bounded geometry.*

Therefore Theorem 1.2.1 follows by considering the isometric injection $\hat{\imath}_{M,f} : M \to \widehat{\mathrm{Cl}}_\infty(\mathrm{im}\,\hat{\imath}_{M,f})$.

There are examples of Lie groups with left invariant metrics that are not coarsely quasi-isometric to any finitely generated group [23], [31]. Applying the above argument to those Riemannian manifolds, we get compact Riemannian foliated spaces whose leaf holonomy covers are not coarsely quasi-isometric to any finitely generated group.

A natural generalization of Theorem 1.2.1 would be given by an affirmative answer of the following.

Question 1.4.6. Let M be a Riemannian manifold of bounded geometry, and G a quotient group of $\pi_1(M)$. Is there some compact Riemannian foliated space having a leaf isometric to (M, g), whose holonomy group is G?

In accordance with the spirit of this thesis, this question can be reduced to the following problem.

Question 1.4.7. Let M' be the Riemannian covering of M corresponding to G. Is there an immersion $f\colon M' \to \mathbb{E}$ of bounded geometry so that $\mathrm{Iso}(M', f) = G$?

Part I

Graph colorings

Chapter 2

Preliminaries on graphs and colorings

2.1 Discrete Metric spaces

Let $X \equiv (X,d)$ be a discrete metric space. For $x \in X$ and $r \geq 0$, let $S(x,r)$ denote the sphere of center x and radius r, and let $B(x,r)$ denote the closed ball of center x and radius r. For another integer $s \geq r \geq 0$, let $C(x,r,s) = B(x,s) \setminus B(x,r)$.

The *penumbra* around any $A \subset X$ of *radius* r is

$$\operatorname{Pen}(A,r) = \{\, x \in X \mid d(x,A) < r \,\} = \bigcup_{y \in A} B(y,r) \ .$$

For a family of subsets $A_i \subset X$, note that

$$\operatorname{Pen}\left(\bigcup_i A_i, r\right) = \bigcup_i \operatorname{Pen}(A_i,r) \ . \tag{2.1}$$

For $r,s \geq 0$, the triangle inequality gives

$$\operatorname{Pen}(\operatorname{Pen}(A,r),s) = \operatorname{Pen}(A,r+s) \ . \tag{2.2}$$

Remark 2.1.1. Note that, for discrete metric spaces, $B(x,r)$ denotes the closed ball or radius r. The exact same notation is used for the open balls when referring to manifolds. The context will make clear if the ambient space is a discrete metric space or a manifold, so that no confusion can arise.

It is said that A is (K-) *separated* (respectively, a (K-) *net*[1]) in X if there is some $K \in \mathbb{N}$ such that $d(x,y) \geq K$ for all $x \neq y$ in A (respectively, for all $x \in X$, there is some $y \in A$ with $d(x,y) \leq K$). We may add X as a subindex to the notation of the balls or crowns if necessary. Recall that a metric space is called *proper* if all of its closed balls are compact (the function $d(x,\cdot) : X \to \mathbb{R}$ is proper for each $x \in X$).

[1]The term *net* was introduced with this meaning by Gromov, but some authors use it for subsets that are also separated.

Lemma 2.1.1. *For $K, L \geq 0$, if A is a K-net in X and B is an L-net in A, then B is a $(K+L)$-net in X.*

Proof. Apply the triangle inequality. □

Lemma 2.1.2. *For $K \geq 0$, if $X = \bigcup_{n=0}^{\infty} A_n$, where $A_0 \subset A_1 \subset \cdots$ and each A_n is K-separated, then X is K-separated.*

Proof. Given $x \neq y$ in X, we have $x, y \in A_n$ for some n, and therefore $d(x,y) \geq K$. □

Lemma 2.1.3. *For $K \geq 0$ and subsets $X_n \subset X$ ($n \in \mathbb{N}$), if $X = \bigcup_{n=0}^{\infty} X_n$ and A_n is a K-net of each X_n, then $A = \bigcup_{n=0}^{\infty} A_n$ is a K-net in X.*

Proof. Any $x \in X$ is contained in some X_n, and therefore $d(x, A_n) \leq K$. □

Lemma 2.1.4. *For $K \geq 0$, any K-separated subset of X is contained in some maximal K-separated K-net of X.*

Proof. Let A be a K-separated subset of X. By Lemma 2.1.2, we can apply Zorn's lemma to the family of K-separated subsets of X containing A, obtaining a maximal K-separated subset B of X. It is easy to check that B is also a K-net (see [7, Lemma 2.1] and [8, Lemma 2.3 and Remark 2.4]). □

Lemma 2.1.5. *A maximal K-separated subset A of X is a $(K-1)$-net.*

Proof. Suppose on the contrary that A is not an $(K-1)$-net in X. Then there must be some point $x \in X$ so that $d(x, A) \geq K$. This implies that the set $A \cup \{x\}$ is a K-separated subset of X containing A, contradicting its maximality. □

2.2 Graphs

An (*undirected*) *graph* $X \equiv (X, E)$ is a set X and a family E of subsets $e \subset X$ with[2] $|e| = 2$. The elements of X and E are called *vertices* and *edges*, respectively. If an edge e contains a vertex x, it is said that e *connects* to x (or e and x are *incident*). The *degree* (or *valency*) $\deg x$ of a vertex x is the number of edges connecting to x. Then let $\deg X = \sup_{x \in X} \deg x < \infty$. Two different vertices are *adjacent* if they define an edge. Two different edges are *consecutive* if they have a common vertex. For[3] $n \in \mathbb{N}$, a *path*[4] of *length* n from x to y in X is a sequence of n consecutive edges joining x to y; in terms of their vertices, it can be considered as a sequence $(z_0, \ldots, z_n)$, where $z_0 = x$, $z_n = y$,

[2] The cardinality of a set X is denoted by $|X|$.
[3] We assume that $0 \in \mathbb{N}$.
[4] For a graph, the topological/metric concepts have their original meaning in the geometric realization.

and z_{i-1} and z_i are adjacent vertices for all $i = 1, \dots, n$. If any two vertices of X can be joined by a path, then X is called *connected*.

Let $X' \equiv (X', E')$ be another graph. An bijection $X \to X'$ is an *isomorphism* (*of graphs*) if it induces a bijection $E \to E'$. When X and X' are connected, a bijection $X \to X'$ is an isomorphism if an only if it is an isometry. Given distinguished points, $x_0 \in X$ and $x'_0 \in X'$, a (*pointed*) *isomorphism* $f : (X, x_0) \to (X', x'_0)$ is an isomorphism $f : X \to X'$ satisfying $f(x_0) = x'_0$. If there is an isomorphism $X \to X'$ (respectively, $(X, x_0) \to (X', x'_0)$), then these structures are called *isomorphic*, and the notation $X \cong X'$ (respectively, $(X, x_0) \cong (X', x'_0)$) may be used. The composition of isomorphisms is another isomorphism. An isomorphism $X \to X$ (respectively, $(X, x_0) \to (X, x_0)$) is called an *automorphism* of X (respectively, (X, x_0)). The group of automorphisms of X (respectively, (X, x_0)) is denoted by $\mathrm{Aut}(X)$ (respectively, $\mathrm{Aut}(X, x_0)$).

We will will make successively stronger assumptions on X. To begin with, assume that X is connected[5]. We get a metric space $X \equiv (X, d)$, where d is the $\mathbb{Z}$-valued metric defined by declaring $d(x, y)$ to be the minimum length of paths in X from x to y. The following property is easily verified:

$$\forall x, y \in X,\ \forall m, n \in \mathbb{N},\ d(x, y) = m + n \Longrightarrow \exists z \in X \mid d(x, z) = m,\ d(y, z) = n\ . \tag{2.3}$$

Note that $E = \{\, \{x, y\} \mid x, y \in X,\ d(x, y) = 1 \,\}$. Therefore E and d are equivalent objects; in fact, this correspondence defines a bijection between the families of connected graph structures and $\mathbb{Z}$-valued metrics satisfying (2.3). Thus an isomorphism between connected graphs is the same as an isometry, and both of these terms will be indistinctly used. All metric concepts of X refer to the induced metric d. Note that each ball $B(x, r)$ in X is connected. Thus A is a K-net if and only if $\mathrm{Pen}(A, K) = X$. More generally, each penumbra $\mathrm{Pen}(A, r)$ in X is connected if A is connected. Note that $|S(x, 0)| = 1$, and $|S(x, 1)|$ is the degree of each vertex x. A path $(u_0, \dots, u_n)$ in X is called a *minimizing geodesic segment* if $d(u_0, u_n) = n$. Using (2.3), it follows that there exist a minimizing geodesic segment joining any pair of vertices.

On any $Y \subset X$, we get the induced graph structure $E|_Y = \{\, \{x, y\} \in E \mid x, y \in Y \,\}$. Then $Y \equiv (Y, E|_Y)$ is called a *subgraph* of X. By Zorn's lemma, there are maximal connected subgraphs of X, called *connected components*, which form a partition of X. Any connected subgraph of X is contained in some connected component of X. If $(Y, E|_Y) \subset (X, E)$ is connected, then Y has two canonical distance functions: one, denoted by d_X, is the restriction to Y of the distance function of the graph (X, E); the second is the distance function induced by the graph structure (Y, E_Y), denoted by d_Y.

[5]This condition is assumed for the sake of simplicity, but it is not essential. The concepts and arguments could be applied to each connected component.

Lemma 2.2.1. *Let X be a graph, $R, T \in \mathbb{N}$, τ be a path of length T between two points $y, z \in B(x, R)$ and $h\colon B(x, R+T) \to B(x', RT)$ be an isomorphism of graphs. Then $\tau \subset \operatorname{dom}(h)$ and τ is a geodesic if and only if $h(\tau)$ is a geodesic.*

Lemma 2.2.2. *Let G be a connected graph. If $A \subset B(x, r)$ is s-separated with respect to d_G, then $A \cap B(x, r-s/2)$ is s-separated with respect to $\overline{d_{B(x,r-s/2)}}$.*

Lemma 2.2.3. *Let G be a connected graph. If $h\colon (B(x, 2r), x) \to (B(y, 2r), y)$ is an isomorphism with respect to the induced subgraph structure, then the restriction of h to $B(x, r)$ preserves the global metric.*

Lemma 2.2.4. *If every vertex of X is adjacent to a countable set of vertices, then X is countable.*

Proof. Given any $x \in X$, since $X = \bigcup_{r=0}^{\infty} S(x, r)$, it is enough to prove that $S(x, r)$ is countable for all $r \in \mathbb{N}$. This is done by induction on r. We have $S(x, 0) = \{x\}$, and $S(x, 1)$ is countable by hypothesis. If $S(x, r)$ is countable for some $r \in \mathbb{N}$, then $S(s, r+1)$ is also countable because it is contained in $\bigcup_{y \in S(x,r)} S(y, 1)$. □

Lemma 2.2.5. *The vertices of X have finite degree if and only if X is a proper metric space.*

Proof. Since d induces the discrete topology, the condition of being proper means that the balls are finite. Then the "if" part is true because $|B(x, 1)| = 1 + \deg x$ for all $x \in X$. Now, assume that the vertices have finite degree, and let us show that $|B(x, r)| < \infty$ for all $x \in X$ and $r \in \mathbb{Z}_+$. This follows by induction on r using that $B(x, r+1) = \operatorname{Pen}(B(x, r), 1)$ by (2.3). □

Lemma 2.2.6. *If $h\colon (B(x, r), x) \to (B(y, r), y)$ is a pointed isomorphism, then h preserves the global metric over $B(x, s)$ for every $s < r/2$.*

Lemma 2.2.7. *Let $x \in X$, $K \in \mathbb{N}$ and $r \in \mathbb{Z}_+$. Let A be a K-net in X. If $r > K$, then $A \cap B(x, r)$ is a $2K$-net in $B(x, r)$.*

Proof. Let $y \in X$. If $y \in B(x, r-K)$, then any $z \in A$ with $d(y, z) \leq K$ is in $B(x, r)$ by the triangle inequality.

If $x \in C(x, r-K, r)$, there is some $z \in B(x, r-K)$ with $d(y, z) \leq K$ by (2.3). Take some $a \in A$ with $d(y, a) \leq K$. Then $a \in B(x_0, r)$ and $d(x, a) \leq 2K$ by the triangle inequality. □

Suppose that X is also unbounded.

Lemma 2.2.8. $|S(x, r)| \geq 1$ *for all $x \in X$ and $r \in \mathbb{N}$.*

Proof. By (2.3) and since X is unbounded, we have $S(x,r) \neq \emptyset$ for all $r \in \mathbb{N}$, yielding $|S(x,r)| \geq 1$. $\square$

Corollary 2.2.9. *$|B(x,r)| \geq r+1$ for all $x \in X$ and $r \in \mathbb{Z}_+$.*

Proof. Apply Lemma 2.2.8 to the disjoint unions $B(x,r) = \bigcup_{i=0}^{r-1} S(x,i)$ and $C(x,r,s) = \bigcup_{i=r}^{s-1} S(x,i)$. $\square$

Lemma 2.2.10. *Let $x \in X$, $K \in \mathbb{N}$ and $r \in \mathbb{Z}_+$. If A is a K-net in $X \setminus B(x,r)$, then A is a $(2r+K-1)$-net in X.*

Proof. By Lemma 2.2.8, there is some $z \in S(x,r)$. Take some $a \in A$ with $d(z,a) \leq K$. Then $d(y,a) < 2r+K$ for all $y \in B(x,r)$ by the triangle inequality. $\square$

Finally, suppose also that there is an least upper bound $\Delta(X) \in \mathbb{N}$ on the vertex degrees of X. For simplicity, we will denote $\Delta(X)$ simply as Δ, and it will be refered to as the *degree* of X. Since X is connected and unbounded, this is only possible for $\Delta \geq 2$ (a connected graph is a singleton if $\Delta = 0$, and it has two vertices if $\Delta = 1$). Note also that, in this case, a subset of X is finite if and only if it is bounded.

Lemma 2.2.11. *$|S(x,r)| \leq \Delta(\Delta-1)^{r-1}$ for all $x \in X$ and $r \in \mathbb{Z}_+$.*

Proof. The vertex x is adjacent with at most k vertices, which form $S(x,1)$. For all $r \in \mathbb{Z}_+$, any $y \in S(x,r)$ is adjacent with at least one vertex in $S(x,r-1)$ by (2.3), and therefore y is adjacent with at most $\Delta - 1$ vertices in $S(x,r+1)$. Then the inequality $|S(x,r)| \leq \Delta(\Delta-1)^{r-1}$ follows easily by induction on r. $\square$

Observe that

$$u \geq v \geq 1 \Longrightarrow \frac{u+1}{v+1} > \frac{u}{2v} \,. \tag{2.4}$$

Corollary 2.2.12. *Let $x \in X$ and $r \in \mathbb{Z}_+$. If $\Delta = 2$, then $|B(x,r)| \leq 2r-1$. If $\Delta \geq 3$, then*

$$|B(x,r)| \leq 1 + \frac{\Delta\left((\Delta-1)^r - 1\right)}{\Delta-2} \,.$$

Proof. Apply Lemma 2.2.11 to the disjoint union $B(x,r) = \bigsqcup_{i=0}^{r} S(x,i)$. $\square$

Let $r \in \mathbb{Z}_+$. If $\Delta \geq 3$, then

$$|B(x,r)| \leq \frac{\Delta(\Delta-1)^{r-1} - 2}{\Delta-2} < 4(\Delta-1)^{r-1} < \Delta^{r+1} \,, \tag{2.5}$$

by Corollary 2.2.12 and (2.4).

Lemma 2.2.13. *Let X be a finite graph and let A be a K-separated $(K-1)$-net. Then $|A| \geq |X|/(\deg X)^K$.*

Proof. We have

$$X \subset \bigcup_{a\in A} B(a, K-1), \quad \text{and} \quad |X| \leq \sum_{a\in A} |B(a, K-1)|.$$

By Corollary 2.2.12 and (2.5), we have $|B(a, K-1)| \leq (\deg X)^{K-1+1}$, and the result follows. $\square$

Definition 2.2.14. Let (A, d_A) and (B, d_B) be metric spaces. For $m \in \mathbb{N}$, a map $f\colon A \to B$ is an *m-short scale isometry* if, for any $x, y \in A$ with $d_A(x,y) \leq m$, we have $d_B(f(x), f(y)) = d_A(x,y)$.

The proof of the following lemma is elementary.

Lemma 2.2.15. *Let (V, E) be a graph with induced metric d_E, and let $f\colon (B(x,n), x) \to (B(y,n), y)$ be a pointed isomorphism for $x, y \in V$ and $n \in \mathbb{N}$. Then, for $0 \leq m \leq n$, the restriction $f\colon B(x, n-m) \to B(y, n-m)$ is an m-short scale isometry with respect to the restrictions of d_E.*

2.3 Colorings

For sets X and F, a *coloring* of X (by *colors* in F) is a map $\phi : X \to F$. The pair (X, ϕ) is called a *colored set*. The sets of colors F will usually be a finite initial segment[6] of $\mathbb{N}$. For $M \in \mathbb{N}$, let $[M] = \{0, \dots, M-1\}$.

Let X be a graph. A coloring of its vertex set, $\phi : X \to F$, is called a (*vertex*) *coloring* of X, and (X, ϕ) is called a *colored graph*. If $x_0 \in Y \subset X$, then the simplified notation $(Y, \phi) = (Y, \phi|_Y)$ and $(Y, x_0, \phi) = (Y, x_0, \phi|_Y)$ will be used. The following concepts for colored graphs are the obvious extensions of their graph versions: (*pointed*) *isomorphisms*, denoted by $f : (X, \phi) \to (X', \phi')$ and $f : (X, x_0, \phi) \to (X', x_0', \phi')$, *isomorphic* (pointed) colored graphs, denoted by $(X, \phi) \cong (X', \phi')$ and $(X, x_0, \phi) \cong (X', x_0', \phi')$, and *automorphism* groups of (pointed) colored graphs, denoted by $\operatorname{Aut}(X, \phi)$ and $\operatorname{Aut}(X, x_0, \phi)$.

Let $\widehat{\mathcal{G}}_*$ be the set[7] of isomorphism classes, $[X, x, \phi]$, of pointed connected colored graphs, (X, x, ϕ), whose vertices have finite degree. For each $R \in \mathbb{Z}_+$, let

$$\widehat{U}_R = \left\{ ([X, x, \phi], [Y, y, \psi]) \in \widehat{\mathcal{G}}_*^2 \mid (B_Y(y, R), y, \psi) \cong (B_X(x, R), x, \phi) \right\}.$$

These sets form a base of entourages of a uniformity on $\widehat{\mathcal{G}}_*$, which is easily seen to be complete. Moreover this uniformity is metrizable because this base is countable.

[6] Recall that a subsets S of an ordered set $(Z, \leq)$ is an *initial segment* if and only if for each $s \in S$ and $z \in Z$, $z \leq s$ implies $z \in Z$.

[7] These graphs are countable (Lemma 2.2.4), and therefore we can assume that their underlying sets are contained in $\mathbb{N}$. In this way, $\widehat{\mathcal{G}}_*$ becomes a well defined set.

Suppose that F is countable. The induced topology is separable because the elements $[X,x,\phi]$, where X is finite, form a countable dense subset. Thus $\widehat{\mathcal{G}}_*$ becomes a Polish space. Note that the *degree map* $\deg : \widehat{\mathcal{G}}_* \to \mathbb{Z}_+$, $[X,x,\phi] \mapsto \deg x$, and the *evaluation map* $\mathrm{ev} : \widehat{\mathcal{G}}_* \to F$, $[X,x,\phi] \mapsto \phi(x)$, are continuous.

For any connected colored graph (X,ϕ), there is a canonical map $\hat{\iota}_{X,\phi} : X \to \widehat{\mathcal{G}}_*$ defined by $\hat{\iota}_{X,\phi}(x) = [X,x,\phi]$. The image $\operatorname{im}\hat{\iota}_{X,\phi}$ has an induced connected colored graph structure, and all of these images form a canonical partition of $\widehat{\mathcal{G}}_*$. It is easy to verify that $\overline{\operatorname{im}\hat{\iota}_{X,\phi}}$ is saturated by the canonical partition. It is said that (X,ϕ) (or ϕ) is *aperiodic* (*by isometries*[8]) if $\mathrm{Aut}(X,\phi) = \{\mathrm{id}_X\}$, which means that $\hat{\iota}_{X,\phi}$ is injective; otherwise, it is said that (X,ϕ) (or ϕ) is *periodic* (*by isometries*). More strongly, (X,ϕ) (or ϕ) is called *limit-aperiodic*[9] (*by isometries*) if (Y,ψ) is aperiodic for all $[Y,y,\psi] \in \overline{\operatorname{im}\hat{\iota}_{X,\phi}}$. On the other hand, (X,ϕ) (or ϕ) is called *repetitive* (*by isometries*) if $\overline{\operatorname{im}\hat{\iota}_{X,\phi}}$ is a minimal set of the canonical partition (it has no smaller closed saturated nonempty subset).

The following result indicates the role plaid by graphs with an upper bound on the vertex degrees, colored by finitely many colors.

Proposition 2.3.1. *$\overline{\operatorname{im}\hat{\iota}_{X,\phi}}$ is compact if and only if $\deg X < \infty$ and $|\operatorname{im}\phi| < \infty$.*

Proof. The "if" part follows using that, if X is of bounded geometry and $\operatorname{im}\phi$ is finite, then, for each $R \in \mathbb{Z}_+$, the pointed colored balls $(B_X(x,R),x,\phi)$, for $x \in X$, represent finitely many isomorphism classes. The "only if" part follows using the continuity of $\deg : \widehat{\mathcal{G}}_* \to \mathbb{Z}_+$ and $\mathrm{ev} : \widehat{\mathcal{G}}_* \to F$. □

If X is finite, the aperiodicity of ϕ is equivalent to its limit-aperiodicity. An aperiodic coloring ϕ of X by finitely many colors can be easily given (see Corollary 2.3.7 below). limit-aperiodic colorings by finite finitely many colors are much more difficult to find when X is infinite. The following lemma will be useful for that purpose.

Lemma 2.3.2. *(X,ϕ) is limit-aperiodic if and only if, for all sequences, x_i, y_i in X and $R_i, S_i \uparrow \infty$ in $\mathbb{Z}_+$, and pointed isomorphisms,*

$$f_i : (B(x_i,R_i),x_i,\phi) \to (B(x_{i+1},R_i),x_{i+1},\phi)\;, \quad h_i : (B(x_i,S_i),x_i,\phi) \to (B(y_i,S_i),y_i,\phi)\;,$$

such that $d(x_i,y_i) + S_i \le R_i$, $f_i(y_i) = y_{i+1}$, and the diagram

$$\begin{array}{ccc} B(x_{i+1},S_{i+1}) & \xrightarrow{h_{i+1}} & B(y_{i+1},S_{i+1}) \\ {\scriptstyle f_i}\big\uparrow & & \big\uparrow{\scriptstyle f_i} \\ B(x_i,S_i) & \xrightarrow{h_i} & B(y_i,S_i) \end{array} \tag{2.6}$$

[8] This part of the terminology will be omitted except when considering also other types of aperiodicity or limit-aperiodicity.

[9] There is no unanimity about this terminology: the terms aperiodic or strongly aperiodic are sometimes used instead of limit-aperiodic, and non-periodic instead of aperiodic.

is commutative, we have that, either $x_i = y_i$ for i large enough, or $\limsup_i d(x_i, y_i) = \infty$.

Proof. This follows easily from the definition of the topology of $\widehat{\mathcal{G}}_*$. $\square$

Remark 2.3.1. In Lemma 2.3.2, the stated property for all bounded sequences x_i, y_i characterizes the aperiodicity of X. Thus the case of unbounded sequences x_i, y_i describes when (Y, ψ) is aperiodic for all $[Y, y, \psi] \in \overline{\operatorname{im} \hat{\iota}_{X,\phi}} \setminus \operatorname{im} \hat{\iota}_{X,\phi}$.

Lemma 2.3.3. *The colored graph (X, ϕ) is repetitive if and only if there is some point $p \in X$ and sequences $R_i, S_i \uparrow \infty$ in $\mathbb{Z}_+$, such that the sets*

$$\boldsymbol{R}_i := \{\, x \in X \mid [B(p, R_i), p, \phi] = [B(x, R_i), x, \phi] \,\}$$

are S_i-nets in X.

Removing the colorings from the notation, we get the Polish space $\mathcal{G}_*$ of isomorphism classes of pointed connected graphs. In this way, we get canonical maps $\iota_X : X \to \mathcal{G}_*$ for connected graphs X, defining a canonical partition of $\mathcal{G}_*$. Then it is said that X is *aperiodic* (*by isometries*) if ι_X is injective, X is *limit-aperiodic* (*by isometries*) if Y is aperiodic for all $[Y, y] \in \overline{\operatorname{im} \iota_X}$, and X is *repetitive* (*by isometries*) if $\overline{\operatorname{im} \iota_X}$ is a minimal set of the canonical partition. Observe also that the forgetful map $\widehat{\mathcal{G}}_* \to \mathcal{G}_*$ is continuous. By Lemma 2.2.5, the space $\mathcal{G}_*$ is a subspace of the Gromov space $\mathcal{M}_*$ of isomorphism classes of pointed proper metric spaces [36], [37, Chapter 3]. There obvious versions of Lemma 2.3.2 and Proposition 2.3.1 in this setting follow by considering a constant coloring.

The following lemma is well known.

Lemma 2.3.4. *If $\deg X \le k < \infty$, then X has a coloring by $k+1$ colors assigning different colors to adjacent vertices.*

Proof. Enumerate the vertices of X in a sequence x_i (using Lemma 2.2.4). Then we proceed by induction on i to define a coloring $\phi : X \to \mathbb{Z}_{k+1}$ satisfying the stated property. Take $\phi(x_0)$ arbitrarily. If $\phi(x_0), \dots, \phi(x_i)$ are defined for some $i \in \mathbb{N}$, then there is some $\phi(x_{i+1}) \in \mathbb{Z}_{k+1}$ such that $\phi(x_{i+1}) \ne \phi(x_j)$ for all $j \le i$ with x_j adjacent to x_{i+1} (see e.g. [16]). $\square$

Corollary 2.3.5. *If $\deg X \le k < \infty$, then there is a coloring ϕ of X by k^2+1 colors such that*

$$\forall x, y \in X,\ 1 \le d(x, y) \le 2 \Longrightarrow \phi(x) \ne \phi(y)\ . \tag{2.7}$$

Proof. Let $E_2 = \{\, \{x, y\} \in E \mid d(x, y) \leq 2 \,\}$. The graph $X_2 := (X, E_2)$ is connected because $E \subset E_2$. Since $|C(x, 1, 3)| \leq k^2$ for all $x \in X$ by Corollary 2.2.12, it follows that k^2 is an upper bound $k \in \mathbb{N}$ on the vertex degrees of X_2. Thus Lemma 2.3.4 gives a map $\phi : X \to \mathbb{Z}_{k^2+1}$ such that $\phi(x) \neq \phi(y)$ if $\{x, y\} \in E_2$, which means that ϕ satisfies (2.7). □

Remark 2.3.2. There is an obvious version of Corollary 2.3.5 using any $r \in \mathbb{Z}_+$ instead of 2.

For our purposes, the interest of the colorings satisfying (2.7) is the following.

Lemma 2.3.6. *If a coloring ϕ of X satisfies* (2.7)*, then the canonical action of* $\mathrm{Aut}(X, \phi)$ *on X is free.*

Proof. Suppose that $f(x) = x$ some $f \in \mathrm{Aut}(X, \phi)$ and $x \in X$; thus f preserves $S(x, r)$ for all $r \in \mathbb{N}$. We prove that f is the identity on each $S(x, r)$ by induction on r. This is true on $S(x, 0) = \{x\}$. Now, suppose that f is the identity on $S(x, r)$ for some $r \in \mathbb{N}$. Hence f preserves $S(y, 1)$ for all $y \in S(x, r)$. But $d(u, v) \leq 2$ for all $u, v \in S(y, 1)$ by the triangle inequality, obtaining that f is the identity on $S(y, 1)$ because ϕ satisfies (2.7). So f is the identity on $\bigcup_{y \in S(x,r)} S(y, 1)$, which contains $S(x, r + 1)$ by (2.3). □

Corollary 2.3.7. *If* $\deg X \leq k < \infty$*, then X admits an aperiodic coloring by $k^2 + 2$ colors.*

Proof. Let $\phi : X \to \{0, \dots, k^2\}$ be a coloring satisfying (2.7), given by Corollary 2.3.5. For any fixed $x_0 \in X$, let $\psi : X \to \{0, \dots, k^2 + 1\}$ be the coloring equal to ϕ on $X \setminus \{x_0\}$ and with $\phi(x_0) = k^2 + 1$. The canonical action of $\mathrm{Aut}(X, \psi)$ on X is free by Lemma 2.3.6, and x_0 is a fixed point. So $\mathrm{Aut}(X, \psi) = \{\mathrm{id}_X\}$. □

Chapter 3

Limit-aperiodic and repetitive colorings of graphs

This chapter is devoted to the proofs of the theorems about colorings of graphs stated in Section 1.1.

3.1 Finitary version of the main theorem

This chapter is devoted to the proof of Theorem 1.1.1. Actually, we will not prove this precise result, but its finitary version, which comprises the following two theorems.

Theorem 3.1.1. *Let X be a connected infinite simple graph with bounded degree $\Delta = \deg X < \infty$, and let ε_n be an increasing sequence of positive integers. Then there are constants δ_n, with δ_n large enough depending only on Δ, ε_m for $m \leq n$, and δ_m for $m < n$, such that there is a coloring ϕ of X by Δ colors, satisfying*

$$\forall x, y \in X,\ \forall n, \quad 0 < d(x,y) < \varepsilon_n \Rightarrow [B(x,\delta_i), x, \psi^N] \neq [B(y,\delta_i), y, \psi^N]. \tag{3.1}$$

Theorem 3.1.2. *Let X and ε_n be like in Theorem 3.1.1, and let $p \in X$ be a distinguished point. Suppose that, for large enough constants $\mathfrak{r}_n$, recursively depending on Δ, on ε_m for $m \leq n$, and on $\mathfrak{r}_m$ and ω_m for $m < n$, the sets*

$$\Omega_n := \{\, x \in X \mid [B(x,\mathfrak{r}_n), x, d_X] = [B(p,\mathfrak{r}_n), p, d_X] \,\} \tag{3.2}$$

are ω_n-nets in X for some constants ω_n. Then, for some large enough positive integers r_n, depending on Δ, on ε_m for $m \leq n$, and on r_m for $m < n$, there is a coloring ϕ by Δ colors satisfying (3.1) *and such that the sets*

$$\widehat{\Omega}_n := \left\{\, x \in X \mid \left[B\left(x, \textstyle\sum_{j=0}^{n} r_j\right), x, \phi\right] = \left[B\left(p, \textstyle\sum_{j=0}^{n} r_j\right), p, \phi\right] \,\right\}$$

are α_n-nets in X for some positive integers α_n.

Theorem 1.1.1 is a trivial consequence of Theorems 3.1.1 and 3.1.2 by the quantitative description of limit-aperiodicity and repetitivity stated in Lemmas 2.3.2 and 2.3.3.

3.2 Constants

Let X be a graph satisfying the conditions of Theorem 3.1.1, and let ε_n be an increasing sequence of positive integers. By induction on $n \in \mathbb{N}$, we are going to define sequences of positive integers, s_n, $\hat{r}_n$, $\hat{r}_n^{\pm}$, $\bar{r}_n$ and $\bar{r}_n^{\pm}$, and sequences of functions, $\bar{\eta}_n, \boldsymbol{R}_n^{\pm}, \boldsymbol{\lambda}_n, \boldsymbol{K}_n, \overline{\boldsymbol{K}}_n \colon \mathbb{N} \to \mathbb{N}$ and $\boldsymbol{\Lambda}_n, \boldsymbol{\Gamma}_n^{\pm}, \boldsymbol{\Delta}_n \colon \mathbb{N}^{n+1} \to \mathbb{N}$. First, set

$$s_0 = 27 + \varepsilon_0, \quad \Delta_{-1} = \deg(X). \tag{3.3}$$

Define $\bar{\eta}_0 \colon \mathbb{N} \to \mathbb{Q}$ as follows:

$$\bar{\eta}_0(a) = \exp_2\left(\left\lfloor \frac{a - \Delta_{-1}^{11}}{\Delta_{-1}^{3}} \right\rfloor\right). \tag{3.4}$$

Let $\hat{r}_0$ be the smallest positive integer such that

$$\bar{\eta}_0\left(\sqrt{\bar{\eta}_0(\hat{r}_0)} - 6\right) > \left(4(\Delta_{-1} - 1)^{\hat{r}_0 s_0^2 (3s_0+1)} + 6\right)^2. \tag{3.5}$$

Note that this is well-defined since there is a double exponential in the left-hand side of the inequality, whereas there is a single exponential on the right-hand side. Observe also that (3.4) and (3.5) yield

$$\hat{r}_0 > 2^{11} \tag{3.6}$$

because $\Delta_{-1} \geq 2$ since X is infinite. Let

$$\bar{r}_0 = \hat{r}_0(3s_0 + 1). \tag{3.7}$$

From (3.5) and the fact that $\bar{\eta}_0$ is an increasing function we get

$$\begin{aligned}\bar{\eta}_0\left(\sqrt{\bar{\eta}_0(\bar{r}_0)} - 6\right) &> \bar{\eta}_0\left(\sqrt{\bar{\eta}_0(\hat{r}_0)} - 6\right) \\ &> \left(4(\Delta_{-1} - 1)^{\hat{r}_0 s_0^2 (3s_0+1)} + 6\right)^2 = \left(4(\Delta_{-1} - 1)^{\bar{r}_0 s_0^2} + 6\right)^2.\end{aligned} \tag{3.8}$$

Define the remaining functions for $n = 0$ as follows:

$$\boldsymbol{R}_0^{-}(a) = 4a - 1, \qquad \boldsymbol{R}_0^{+}(a) = a(2s_0 + 3), \qquad \boldsymbol{\lambda}_0(a) = 2\boldsymbol{R}_0^{+}(a) + 1, \tag{3.9}$$

$$\boldsymbol{\Delta}_0(a) = 4(\Delta_{-1} - 1)^{2\boldsymbol{R}_0^{+}(a)}, \qquad \boldsymbol{\Lambda}_0(a) = \boldsymbol{\lambda}_0(a), \qquad \boldsymbol{\Gamma}_0^{\pm}(a) = \boldsymbol{R}_0^{\pm}(a). \tag{3.10}$$

Now, given $n > 0$, suppose that we have defined the desired constants and functions for integers $0 \leq m < n$. Let $\bar{\boldsymbol{r}}_{n-1}$ denote the n-tuple $(\bar{r}_0, \dots, \bar{r}_{n-1})$. Then define

$$s_n = 27 + 10\boldsymbol{\Lambda}_{n-1}(\bar{\boldsymbol{r}}_{n-1}) + 2\boldsymbol{\Gamma}_{n-1}^{+}(\bar{\boldsymbol{r}}_{n-1}) + \varepsilon_n. \tag{3.11}$$

Let $\bar{\eta}_n \colon \mathbb{N} \to \mathbb{Q}$ be defined by

$$\bar{\eta}_n(a) = \exp_2\left(\left\lfloor \frac{a - \boldsymbol{\Delta}_{n-1}^{8}(\bar{\boldsymbol{r}}_{n-1})}{\boldsymbol{\Delta}_{n-1}^{3}(\bar{\boldsymbol{r}}_{n-1})} \right\rfloor\right). \tag{3.12}$$

Then, let $\hat{r}_n$ be the smallest positive integer so that

$$\bar{\eta}_n\left(\sqrt{\bar{\eta}_n(\hat{r}_n)}-6\right) > \left(4(\boldsymbol{\Delta}_{n-1}(\bar{\boldsymbol{r}}_{n-1})-1)^{\hat{r}_n s_n^2(3s_n+1)}+6\right)^2. \tag{3.13}$$

This is well-defined like in the case of $\hat{r}_0$. Let

$$\bar{r}_n = \hat{r}_n(3s_n+1). \tag{3.14}$$

From (3.5) and the fact that $\bar{\eta}_n$ is an increasing function we get

$$\begin{aligned}\bar{\eta}_n\left(\sqrt{\bar{\eta}_n(\bar{r}_n)}-6\right) > \bar{\eta}_n\left(\sqrt{\bar{\eta}_n(\hat{r}_n)}-6\right) &> \left(4(\boldsymbol{\Delta}_{n-1}(\bar{\boldsymbol{r}}_{n-1})-1)^{\hat{r}_n s_n^2(3s_n+1)}+6\right)^2 \\ &= \left(4(\boldsymbol{\Delta}_{n-1}(\bar{\boldsymbol{r}}_{n-1})-1)^{\bar{r}_n s_n^2}+6\right)^2.\end{aligned} \tag{3.15}$$

For $n\in\mathbb{N}$, let $\boldsymbol{a}_n$ and $\boldsymbol{a}_{n-1}$ denote the $(n+1)$ and n-tuples $(a_0,\dots,a_n)$ and $(a_0,\dots,a_{n-1})$. Let

$$\boldsymbol{R}_n^-(a) = 4a-1, \quad \boldsymbol{R}_n^+(a) = a(2s_n+3), \quad \boldsymbol{\lambda}_n(a) = 2\boldsymbol{R}_n^+(a)+1, \tag{3.16}$$

$$\boldsymbol{\Delta}_n(\boldsymbol{a}_n) = 4\left(\boldsymbol{\Delta}_{n-1}(\boldsymbol{a}_{n-1})-1\right)^{2\boldsymbol{R}_n^+(\boldsymbol{a}_n)}, \tag{3.17}$$

$$\boldsymbol{\Lambda}_n(\boldsymbol{a}_N) = \prod_{i=0}^{n}\boldsymbol{\lambda}_i(a_i), \quad \boldsymbol{\Gamma}_n^\pm(\boldsymbol{a}_N) = \boldsymbol{R}_n^\pm(a_n)\cdot\boldsymbol{\Lambda}_{n-1}(\boldsymbol{a}_{N-1})+\boldsymbol{\Gamma}_{n-1}^+(\boldsymbol{a}_{N-1}). \tag{3.18}$$

Note that $\boldsymbol{R}_n^-$ is independent of n. Also, by a simple induction argument, we get, for $l=0,\dots,N$,

$$\boldsymbol{\Gamma}_n^\pm(\boldsymbol{a}_N) \geq \boldsymbol{R}_l^\pm(a_l). \tag{3.19}$$

Lemma 3.2.1. *Let $n\in\mathbb{N}$, and let $\boldsymbol{a}=(a_0,\dots,a_n)$ be an $(n+1)$-tuple such that, for $0\leq m\leq n$, we have $a_m\leq\bar{r}_m$. Then*

$$a_n s_n \geq 2\boldsymbol{\Gamma}_n^-(\boldsymbol{a}_n)+\varepsilon_n, \quad a_n s_n^2 \geq 2\boldsymbol{\Gamma}_n^+(\boldsymbol{a}_n)+\varepsilon_n.$$

Proof. By definition of s_n, we have

$$a_n s_n = a_n(10\boldsymbol{\Lambda}_{n-1}(\bar{\boldsymbol{r}}_{n-1})+2\boldsymbol{\Gamma}_{n-1}^+(\bar{\boldsymbol{r}}_{n-1})+\varepsilon_n) > 10a_n\boldsymbol{\Lambda}(\bar{\boldsymbol{r}}_{n-1})+2\boldsymbol{\Gamma}_{n-1}^+(\bar{\boldsymbol{r}}_{n-1})+\varepsilon_n.$$

On the other hand, using (3.16) and the fact that $\boldsymbol{\Lambda}_{n-1}$ and $\boldsymbol{\Gamma}_{n-1}^\pm$ are monotone increasing functions on every coordinate, we have

$$\boldsymbol{\Gamma}_n^\pm(\boldsymbol{a}_n) \leq \boldsymbol{R}_n^\pm(a_n)\cdot\boldsymbol{\Lambda}_{n-1}(\bar{\boldsymbol{r}}_{n-1})+\boldsymbol{\Gamma}_{n-1}^+(\bar{\boldsymbol{r}}_{n-1}).$$

Then the proof follows by showing that $10a_n > 2\boldsymbol{R}_n^-(a_n)$ and $10a_n s_n > 2\boldsymbol{R}_n^+(a_n)$, which is an easy consequence of the definitions. □

Let $\overline{\mathbf{K}}_{-1} = \mathbf{K}_{-1} \equiv \overline{K}_{-1} = K_{-1} = 0$, and continue defining $\overline{\mathbf{K}}_n$ and $\mathbf{K}_n$ by induction on $n \in \mathbb{N}$ as follows:

$$\overline{\boldsymbol{K}}_n(\boldsymbol{a}_n) = \boldsymbol{K}_{n-1}(\boldsymbol{a}_{n-1}) + \boldsymbol{\Lambda}_n(\boldsymbol{a}_n)(a_n s_n^2 + a_n(2s_n + 1)), \tag{3.20}$$

$$\boldsymbol{K}_n(\boldsymbol{a}_n) = \overline{\boldsymbol{K}}_n(\boldsymbol{a}_n) + \boldsymbol{\Lambda}_n(\boldsymbol{a}_n)(s_{n+1}\boldsymbol{R}^+_{n+1}(\bar{\boldsymbol{r}}_{n+1}) + \boldsymbol{\Gamma}^+_n(\bar{\boldsymbol{r}}_n) + 2\boldsymbol{R}^+_n(\bar{\boldsymbol{r}}_n)). \tag{3.21}$$

Finally, for all $n \in \mathbb{N}$, let

$$\bar{r}_n^- = \bar{r}_n \ , \quad \bar{r}_n^+ = s_n\bar{r}_n \ , \quad \hat{r}_n^- = \hat{r}_n \ , \quad \hat{r}_n^+ = s_n\hat{r}_n.$$

3.3 Construction of $\mathfrak{X}_n$

This section is devoted to the construction of subsets $\mathfrak{X}_n \subset X$, which will be used later to achieve the repetitiveness of ϕ under the assumptions of Theorem 3.1.2. Hence we suppose that X satisfies the hypothesis of Theorem 3.1.2 throughout this section. Therefore we have a distinguished point $p \in X$, and, for $n \in \mathbb{N}$, the set

$$\Omega_n = \{\, x \in X \mid [B(p, \mathfrak{r}_n), p, d_X] = [B(x, \mathfrak{r}_n), x, d_X]\,\}$$

is an ω_n-net in X.

For notational convenience, define $\mathfrak{r}_{-1} = \mathfrak{s}_{-1} = \mathfrak{t}_{-1} = \omega_{-1} = 0$. Take increasing sequences of constants $\mathfrak{r}_0, \mathfrak{s}_n, \mathfrak{t}_n > 0$ satisfying the following conditions:

$$\mathfrak{r}_n > \boldsymbol{K}_n(\bar{\boldsymbol{r}}_n) + s_n^2 4\boldsymbol{\Lambda}_n(\bar{\boldsymbol{r}}_n)(\boldsymbol{\Gamma}^+_n(\bar{\boldsymbol{r}}_n) + n), \mathfrak{t}_{n-1} + 2\omega_{n-1} + 1, \tag{3.22}$$

$$\mathfrak{s}_n > 2\mathfrak{r}_n + 2\mathfrak{s}_{n-1}, \boldsymbol{\Lambda}_{n-1}(\bar{\boldsymbol{r}}_{n-1})(2\mathfrak{r}_n + \boldsymbol{K}_{n-1}(\bar{\boldsymbol{r}}_{n-1})), 3\boldsymbol{\Lambda}_n(\bar{\boldsymbol{r}}_n)\boldsymbol{\Gamma}^+_{n+1}(\bar{\boldsymbol{r}}_{n+1}), \tag{3.23}$$

$$\mathfrak{t}_n > \boldsymbol{K}_n(\bar{\boldsymbol{r}}_n), 5\mathfrak{t}_{n-1} + \mathfrak{r}_n + \mathfrak{s}_{n-1} + 2\omega_{n-1} + 1. \tag{3.24}$$

Then, for all $x \in X$ and $0 \le n < m$, define the following subsets of X:

$$\mathfrak{B}^m_n(x) = B(x, \mathfrak{r}_m + \mathfrak{s}_n), \quad \mathfrak{V}^m_n(x) = B(x, \mathfrak{r}_m - \mathfrak{t}_n), \quad \mathfrak{C}^m_n(x) = \mathfrak{B}^m_n(x) \setminus \mathfrak{V}^m_n(x). \tag{3.25}$$

Then the following lemma follows from the above inequalities:

Lemma 3.3.1. *For integers $0 \le n < m$, we have:*

(i) *if $d(x, y) \ge \mathfrak{s}_n$, then $B(x, \mathfrak{r}_n) \cap B(y, \mathfrak{r}_n) = \emptyset$, and $\mathfrak{B}^n_l(x) \cap \mathfrak{B}^n_l(y) = \emptyset$ for $0 \le l < n$;*

(ii) *if $d(x, y) \ge \mathfrak{r}_m + \mathfrak{s}_n$, then $B(x, \mathfrak{r}_m) \cap B(y, \mathfrak{r}_n) = \emptyset$, and $\mathfrak{B}^m_l(x) \cap \mathfrak{B}^n_l(y) = \emptyset$ for $0 \le l < n$; and,*

(iii) *if $d(x, y) \le \mathfrak{r}_m - \mathfrak{t}_n$, then $B(y, \mathfrak{r}_n) \subset B(x, \mathfrak{r}_m)$, and $\mathfrak{B}^n_l(y) \subset B(x, \mathfrak{r}_m - \mathfrak{t}_l - 2\omega_l)$ for $0 \le l < n$.*

For $n \in \mathbb{N}$, define $\mathfrak{Z}_n^n = \{p\}$ and $\mathfrak{f}_{n,p}^n = \mathrm{id}_{B(x,\mathfrak{r}_n)}$. In Proposition 3.3.2, we will continue defining subsets $\mathfrak{Z}_n^m \subset X$ for $0 \le n < m$, and pointed isometries $\mathfrak{f}_{n,z}^m \colon (B(p,\mathfrak{r}_n),p) \to (B(z,\mathfrak{r}_n),z)$ for $z \in \mathfrak{Z}_n^m$. We will use the following notation:

$$\mathfrak{P}_n^m = \{\, (l,z) \in \mathbb{N} \times X \mid n < l < m,\ z \in \mathfrak{Z}_l^m \,\}, \tag{3.26}$$

$$\mathfrak{Q}_n^m = \bigcup_{(l,z)\in\mathfrak{P}_n^m} \mathfrak{C}_n^l(z). \tag{3.27}$$

Remark 3.3.1. Note that, given integers $0 \le n < m$, the definitions of $\mathfrak{P}_n^m$ and $\mathfrak{Q}_n^m$ only make reference to sets $\mathfrak{Z}_k^l$ when either $l < m$ and $k \ge n$, or $l = m$ and $k > n$.

Let $<$ denote the binary relation on $\mathfrak{P}_n^m$ defined by declaring $(l,z) < (l',z')$ if $l < l'$ and $B(z,\mathfrak{r}_l) \subset B(z',\mathfrak{r}_{l'})$, and let $\le$ denote the reflexive closure of $<$.

Proposition 3.3.2. *For $0 \le n < m$, there are sets $\mathfrak{Z}_n^m \subset X$, and for each $z \in \mathfrak{Z}_n^m$ there is a pointed isometry $\mathfrak{f}_{n,z}^m \colon (B(p,\mathfrak{r}_n),p) \to (B(z,\mathfrak{r}_n),z)$, satisfying the following properties:*

(i) *The set $\mathfrak{Z}_n^m$ is a maximal $\mathfrak{s}_n$-separated subset of $\Omega_n \cap \mathfrak{V}_n^m(p) \setminus \mathfrak{Q}_n^m$.*

(ii) *For any $x \in \mathfrak{Z}_n^m$ and $(l,z) \in \mathfrak{P}_n^m$,*

  (a) *either $x \notin \mathfrak{B}_n^l(z)$ and, for every $0 \le l' < n$, we have $\mathfrak{B}_{l'}^n(x) \cap \mathfrak{B}_{l'}^l(z) = \emptyset$, or*

  (b) *$x \in \mathfrak{V}_n^l(z)$ and, for every $0 \le l' < n$, we have $\mathfrak{B}_{l'}^n(x) \subset \mathfrak{V}_{l'}^l(z)$.*

(iii) *For any $(l,z) \in \mathfrak{P}_n^m$, one has $\mathfrak{Z}_n^m \cap \mathfrak{B}_n^l(z) = \mathfrak{f}_{l,z}^m(\mathfrak{Z}_n^l)$.*

(iv) *For any $x \in \mathfrak{Z}_n^m$ and $(l,z) \in \mathfrak{P}_n^m$ such that* (b) *holds, we have $\mathfrak{f}_{n,x}^m = \mathfrak{f}_{l,z}^m \circ \mathfrak{f}_{n,x'}^l$ on $B(p,\mathfrak{r}_n)$, where $x' = (\mathfrak{f}_{l,z}^m)^{-1}(x)$.*

(v) *Consider integers $0 \le k \le l$ such that either $l < m$ and $k \ge n$, or $l = m$ and $k > n$. Then $\mathfrak{Z}_k^l \subset \mathfrak{Z}_n^m$, and for any $z \in \mathfrak{Z}_k^l$ we have $\mathfrak{f}_{n,z}^m = \mathfrak{f}_{k,z}^l|_{B(p,\mathfrak{r}_n)}$.*

(vi) *We have $p \in \mathfrak{Z}_n^m$ and $\mathfrak{f}_{n,p}^m = \mathrm{id}_{B(p,\mathfrak{r}_n)}$.*

Remark 3.3.2. In (iv), the equality $\mathfrak{f}_{n,x}^m = \mathfrak{f}_{l,z}^m \circ \mathfrak{f}_{n,x'}^l$ makes sense on $B(p,\mathfrak{r}_n)$ because $B(x',\mathfrak{r}_n) \subset B(p,\mathfrak{r}_l)$ or, equivalently, $B(x,\mathfrak{r}_n) \subset B(z,\mathfrak{r}_l)$. This holds since, for all $y \in B(x,\mathfrak{r}_n)$, we have $d(y,z) \le d(y,x) + d(x,z) < \mathfrak{r}_n + \mathfrak{r}_l - \mathfrak{t}_n < \mathfrak{r}_l$ by (b) and (3.24).

Proof. First, note that, for integers $0 \le n < m$, we can see using Remark 3.3.1 that properties (i) (v) only reference points $z \in \mathfrak{Z}_k^l$ or isometries $\mathfrak{f}_{k,z}^l$ where either $l < m$, or $l = m$ and $k \ge n$. This allows us to proceed inductively in the following way. First we define for $n \ge 1$, the sets $\mathfrak{Z}_{n-1}^n$ and for each point $z \in \mathfrak{Z}_{n-1}^n$, the isometries $\mathfrak{f}_{n-1,z}^n$. Then we construct, for $0 \le n < m-1$, the sets $\mathfrak{Z}_n^m$, and, for each point $z \in \mathfrak{Z}_n^m$, the isometries

$\mathfrak{f}^n_{m,z}$, under the assumption that we have already defined $\mathfrak{Z}^l_k$ and $\mathfrak{f}^l_{k,z}$ when either $l < m$, or $l = m$ and $k > n$.

For $n \geq 1$, let $\mathfrak{Z}^n_{n-1}$ be any maximal $\mathfrak{s}_{n-1}$-separated subset of $\Omega_{n-1} \cap \mathfrak{W}^n_{n-1}(p)$ containing p. Then define $\mathfrak{f}^n_{n-1,p} = \mathrm{id}_{B(p,\mathfrak{r}_{n-1})}$ and, for each $z \in \mathfrak{Z}^n_{n-1}$, let $\mathfrak{f}^n_{n-1,z}$ be any pointed isometry $(B(p,\mathfrak{r}_{n-1}),p) \to (B(z,\mathfrak{r}_{n-1}),z)$, which exists by the assumption that $z \in \Omega_{n-1}$. The fact that these definitions satisfy properties (i)–(v) follows easily after realizing that $\mathfrak{P}^n_{n-1} = \emptyset$.

As induction hypothesis, suppose now that, given $0 \leq n < m$, we have already defined $\mathfrak{Z}^l_k$ and $\mathfrak{f}^l_{k,z}$ for $l < m$, or $l = m$ and $k > n$.

Claim 3.3.1. (a) For any $(l,z) \in \mathfrak{P}^m_n$, we have $\mathfrak{B}^l_n(z) \subset B(p,\mathfrak{r}_m)$.

(b) We have $\mathfrak{Q}^m_n \subset B(p,\mathfrak{r}_m)$.

By the induction hypothesis with (i), we have $d(p,z) \leq \mathfrak{r}_m - \mathfrak{t}_l$, and by (3.25) we have $\mathfrak{B}^l_n(z) = B(z,\mathfrak{r}_l + \mathfrak{s}_n)$. Now, by (3.23) and the triangle inequality, we get (a). Then (b) follows from (3.27), completing the proof of Claim 3.3.1.

Claim 3.3.2. (a) Let $(l,z),(l,z') \in \mathfrak{P}^m_n$ satisfy at least one of the following properties:

(i) $(l,z) \leq (l,z')$;

(ii) $\mathfrak{B}^l_n(z) \cap \mathfrak{B}^l_n(z') \neq \emptyset$; or

(iii) $d(z,z') < \mathfrak{s}_l$.

Then it holds that $z = z'$.

(b) $(\mathfrak{P}^m_n, \leq)$ is a partially ordered set.

Let us prove (a). From the definition of $\mathfrak{P}^m_n$ in (3.26), we see that $(l,z) \in \mathfrak{P}^m_n$ implies $l > n$. Any of the properties (i) or (ii) implies that $d(z,z') \leq 2\mathfrak{r}_l + 2\mathfrak{s}_n$. Using now (3.23), we infer that $d(z,z') < \mathfrak{s}_l$. So any of (i) or (ii) implies (iii). By the induction hypothesis, the set $\mathfrak{Z}^m_l$ is $\mathfrak{s}_l$-separated, obtaining $z = z'$.

Let us prove (b). The relation $\leq$ is reflexive because it was defined as the reflexive closure of $<$. It is trivial to check that it is also transitive using the definition of $<$. Thus it only remains to prove that it is also antisymmetric. Let $(l,z),(l',z') \in \mathfrak{P}^m_n$ such that $(l,z) \leq (l',z')$ and $(l',z') \leq (l,z)$. By the definition of $\leq$, we get $l = l'$. But now we can apply (a) and conclude that $z = z'$, and therefore $(l,z) = (l',z')$.

Let $\overline{\mathfrak{P}}^m_n$ denote the set of maximal elements of $(\mathfrak{P}^m_n, \leq)$.

Claim 3.3.3. (a) For every $(l,z),(l',z') \in \mathfrak{P}^m_n$ such that $\mathfrak{B}^l_n(z) \cap \mathfrak{B}^{l'}_n(z') \neq \emptyset$, we have that $(l,z) \leq (l',z')$ or $(l',z') \leq (l,z)$.

(b) For every $x \in \bigcup_{(l,z)\in\mathfrak{P}_n^m} \mathfrak{B}_n^l(z)$ (respectively, $(l',z') \in \mathfrak{P}_n^m$) , there is a unique $(l,z) \in \overline{\mathfrak{P}}_n^m$ such that $x \in \mathfrak{B}_n^l(z)$ (respectively, $(l',z') \le (l,z)$).

(c) For every $(l,z),(l',z') \in \mathfrak{P}_n^m$, the following conditions are equivalent:

- $(l',z') \le (l,z)$;
- $\mathfrak{C}_n^{l'}(z') \cap \mathfrak{B}_n^l(z) \neq \emptyset$;
- $\mathfrak{C}_n^{l'}(z') \subset \mathfrak{V}_n^l(z)$.

(d) For all $(l,z),(l',z') \in \mathfrak{P}_n^m$ with $(l,z) \neq (l',z')$, we have that $\mathfrak{C}_n^l(z) \cap \mathfrak{C}_n^{l'}(z') = \emptyset$.

(e) For every $(l,z) \in \mathfrak{P}_n^m$, we have:

$$\left\{\, \left(l', \mathfrak{f}_{l,z}^m(z')\right) \mid (l',z') \in \mathfrak{P}_n^l \,\right\} = \{\, (l',z') \in \mathfrak{P}_n^m \mid (l',z') < (l,z) \,\}\,, \tag{3.28}$$

$$\mathfrak{f}_{l,z}^m\left(\mathfrak{Q}_n^l\right) = \bigsqcup_{(l',z')<(l,z)} \mathfrak{C}_n^{l'}(z'), \tag{3.29}$$

$$\mathfrak{Q}_n^m \cap \mathfrak{B}_n^l(z) = \mathfrak{C}_n^l(z) \sqcup \mathfrak{f}_{l,z}^m\left(\mathfrak{Q}_n^l\right), \tag{3.30}$$

$$\begin{aligned}\mathfrak{V}_n^m(p) \setminus \mathfrak{Q}_n^m = &\left(\bigsqcup_{(l,z)\in\overline{\mathfrak{P}}_n^m} \mathfrak{V}_n^l(z) \setminus \mathfrak{f}_{l,z}^m\left(\mathfrak{Q}_n^l\right) \right) \\ &\sqcup \left(\mathfrak{V}_n^m(p) \setminus \bigsqcup_{(l',z')\in\overline{\mathfrak{P}}_n^m} \mathfrak{B}_n^{l'}(z') \right).\end{aligned} \tag{3.31}$$

Let us prove (a). The case where $l = l'$ follows immediately from Claim 3.3.2 (a). Suppose now that $l' > l$. Then, by the induction hypothesis and using Property 3.3.2 (ii), we get that either $\mathfrak{B}_n^l(z) \cap \mathfrak{B}_n^{l'}(z') = \emptyset$ or $\mathfrak{B}_n^l(z) \subset \mathfrak{V}_n^{l'}(z')$. But, by assumption, $\mathfrak{B}_n^l(z) \cap \mathfrak{B}_n^{l'}(z') \neq \emptyset$. So then $\mathfrak{B}_n^l(z) \subset \mathfrak{V}_n^{l'}(z')$ and therefore $(l,z) \le (l',z')$. The case where $l' < l$ is analogous.

Property (b) follows immediatly from (a) since chains in $\mathfrak{P}_n^m$ are finite because we take $n < l < m$.

Let us prove (c). We prove first that the condition $(l',z') \le (l,z)$ is equivalent to $\mathfrak{C}_n^{l'}(z) \cap \mathfrak{B}_n^l(z) \neq \emptyset$. If $l' = l$, this equivalence follows from Claim 3.3.2 (a). Suppose now that $l' < l$. Using Proposition 3.3.2 (ii) and the induction hypothesis, we see that the condition $B(z', \mathfrak{r}_{l'}) \subset B(z, \mathfrak{r}_l)$ is equivalent to $\mathfrak{B}_n^{l'}(z') \subset \mathfrak{V}_n^l(z)$, which in turn is equivalent to $\mathfrak{C}_n^{l'}(z') \cap \mathfrak{V}_n^l(z) \neq \emptyset$ because $\mathfrak{C}_n^{l'}(z') \neq \emptyset$ since X is infinite. In the last case where $l' > l$, we clearly have that $(l',z') \not\le (l,z)$. Property (ii) implies that either $\mathfrak{B}_n^l(z) \cap \mathfrak{B}_n^{l'}(z') = \emptyset$ or $\mathfrak{B}_n^l(z) \subset \mathfrak{V}_n^{l'}(z')$, and we have $\mathfrak{C}_n^{l'}(z') \cap \mathfrak{V}_n^l(z) = \emptyset$ in either case.

The fact that $\mathfrak{C}_n^{l'}(z) \cap \mathfrak{B}_n^l(z) \neq \emptyset$ is equivalent to $\mathfrak{C}_n^{l'}(z) \subset \mathfrak{V}_n^l(z)$ follows from the induction hypothesis using property (b).

Property (d) is just an immediate consequence of (c).

Let us prove (e). Let $(l',z') \in \mathfrak{P}^l_n$. By definition, this means that $n < l' < l$ and $z' \in \mathfrak{Z}^l_{l'}$, which in turn is equivalent to the condition that $n < l' < l$ and $\mathfrak{f}^m_{l,z}(z') \in \mathfrak{f}^m_{l,z}(\mathfrak{Z}^l_{l'})$. Using now Proposition 3.3.2 (iii) and the inducion hypothesis, we obtain that this is equivalent to the fact that $n < l < l'$ and $\mathfrak{f}^m_{l,z}(z') \in \mathfrak{Z}^m_{l'} \cap \mathfrak{B}^l_n(z)$. By Proposition 3.3.2 (ii), this is equivalent to the condition that $n < l < l'$ and $B(\mathfrak{f}^m_{l,z}(z'), \mathfrak{r}_{l'}) \subset B(z, \mathfrak{r}_n)$, which defines the set of pairs $(l'',z'') \in \mathfrak{P}^m_n$ such that $(l'',z'') < (l,z)$, completing the proof of equation 3.28.

Let $(l',z') \in \mathfrak{P}^l_n$. By definition, $z' \in \mathfrak{Z}^l_{l'}$, and therefore $z' \in \mathfrak{V}^l_{l'}(p)$ by the induction hypothesis with Proposition 3.3.2 (i). Then, by Claim 3.3.1 (b), we have

$$\mathfrak{C}^{l'}_n(z) \subset \mathfrak{Q}^m_n \subset B(p, \mathfrak{r}_l) = \operatorname{dom}(\mathfrak{f}^m_{l,z}). \tag{3.32}$$

A consequence of (a) is that, for any two elements $(l,z),(l',z') \in \mathfrak{P}^m_n$ with $(l,z) \neq (l',z')$, we have $\mathfrak{B}^l_n(z) \cap \mathfrak{B}^{l'}_n(z') = \emptyset$. Using the definition of $\mathfrak{Q}^l_n$, Claim 3.3.3 (d) and (3.32) we obtain

$$\mathfrak{f}^m_{l,z}(\mathfrak{Q}^l_n) = \mathfrak{f}^m_{l,z}\left(\bigsqcup_{(l',z')\in\mathfrak{P}^l_n} \mathfrak{C}^{l'}_n(z')\right).$$

But $\mathfrak{f}^m_{l,z}$ is an isometry, and therefore $\mathfrak{f}^m_{l,z}\left(\mathfrak{C}^{l'}_n(z')\right) = \mathfrak{C}^{l'}_n(\mathfrak{f}^m_{l,z}(z'))$, yielding

$$\mathfrak{f}^m_{l,z}(\mathfrak{Q}^l_n) = \bigsqcup_{(l',z')\in\mathfrak{P}^l_n} \mathfrak{C}^{l'}_n\left(\mathfrak{f}^m_{l,z}(z')\right).$$

Using now equality 3.28, we obtain 3.29.

From Claim 3.3.3 (d) we get that $\mathfrak{Q}^m_n = \bigsqcup_{(l,z)\in\mathfrak{P}^m_n} \mathfrak{C}^l_n(z)$. Then we have the obvious equality:

$$\mathfrak{Q}^m_n \cap \mathfrak{B}^l_n(z) = \bigsqcup_{(l',z')\in\mathfrak{P}^m_n,\, \mathfrak{C}^{l'}_n(z')\cap\mathfrak{B}^l_n(z)\neq\emptyset} \mathfrak{C}^{l'}_n(z') \cap \mathfrak{B}^l_n(z).$$

Using (c) we conclude the following:

$$\mathfrak{Q}^m_n \cap \mathfrak{B}^l_n(z) = \bigsqcup_{(l',z')\leq(l,z)} \mathfrak{C}^{l'}_n(z') = \mathfrak{C}^l_n(z) \sqcup \bigsqcup_{(l',z')<(l,z)} \mathfrak{C}^{l'}_n(z').$$

So applying 3.29 we get 3.30. The induction hypothesis with (i) and Lemma 3.3.1 (iii) yield $\mathfrak{B}^l_n(z) \subset \mathfrak{V}^m_n(p)$ for all $(l,z) \in \overline{\mathfrak{P}}^m_n$, and therefore

$$\mathfrak{V}^m_n(p) = \left(\mathfrak{V}^m_n(p) \setminus \bigsqcup_{(l,z)\in\overline{\mathfrak{P}}^m_n} \mathfrak{B}^l_n(z)\right) \sqcup \bigsqcup_{(l,z)\in\overline{\mathfrak{P}}^m_n} \mathfrak{B}^l_n(z).$$

We have established that for any two elements $(l,z),(l',z') \in \mathfrak{P}^m_n$ with $(l,z) \neq (l',z')$, we have $\mathfrak{B}^l_n(z) \cap \mathfrak{B}^{l'}_n(z') = \emptyset$. It is clear that for any $(l,z) \in \overline{\mathfrak{P}}^m_n$, we have $\mathfrak{C}^l_n(z) \subset$

$\bigsqcup_{(l',z')\in\overline{\mathfrak{P}}_n^m}\mathfrak{B}_n^{l'}(z')$. Thus $\mathfrak{Q}_n^m\subset\bigsqcup_{(l,z)\in\overline{\mathfrak{P}}_n^m}\mathfrak{B}_n^l(z)$ by its definition in (3.27), and therefore:

$$\mathfrak{V}_n^m(p)\setminus\mathfrak{Q}_n^m=\left(\mathfrak{V}_n^m(p)\setminus\bigsqcup_{(l,z)\in\overline{\mathfrak{P}}_n^m}\mathfrak{B}_n^l(z)\right)\sqcup\left(\bigsqcup_{(l,z)\in\overline{\mathfrak{P}}_n^m}\mathfrak{B}_n^l(z)\setminus\mathfrak{Q}_n^m\right).$$

Then, to complete the proof of 3.31, it is enough to prove that, for every $(l,z)\in\overline{\mathfrak{P}}_n^m$, we have $\mathfrak{B}_n^l(z)\setminus\mathfrak{Q}_n^m=\mathfrak{V}_n^l(z)\setminus\mathfrak{f}_{l,z}^m(\mathfrak{Q}_n^l)$. By 3.30, we obtain $\mathfrak{B}_n^l(z)\setminus\mathfrak{Q}_n^m=\mathfrak{B}_n^l(z)\setminus(\mathfrak{C}_n^l(z)\sqcup\mathfrak{f}_{l,z}^m(\mathfrak{Q}_n^l))$. But, by definition, we get $\mathfrak{B}_n^l(z)\setminus\mathfrak{C}_n^l(z)=\mathfrak{V}_n^l(z)$, and finally we obtain $\mathfrak{B}_n^l(z)\setminus\mathfrak{Q}_n^m=\mathfrak{V}_n^l(z)\setminus\mathfrak{f}_{l,z}^m(\mathfrak{Q}_n^l)$, completing the last part of the proof Claim 3.3.3.

Let us define $\mathfrak{Z}_n^m$. First, let

$$\widetilde{\mathfrak{Z}}_n^m=\bigcup_{(l,z)\in\overline{\mathfrak{P}}_n^m}\mathfrak{f}_{l,z}^m\left(\mathfrak{Z}_n^l\right).$$

Note that this is well defined since by the induction hypothesis with (i) we have

$$\mathfrak{Z}_l^m\subset\mathfrak{V}_l^m(p)\subset B(p,\mathfrak{r}_l)=\operatorname{dom}(\mathfrak{f}_{l,z}^m). \tag{3.33}$$

Second, let $\widehat{\mathfrak{Z}}_n^m$ be any maximal $\mathfrak{s}_n$-separated subset of

$$\Omega_n\cap\mathfrak{V}_n^m(p)\setminus\bigsqcup_{(l,z)\in\overline{\mathfrak{P}}_n^m}\mathfrak{B}_n^l(z).$$

We have $\widetilde{\mathfrak{Z}}_n^m\cap\widehat{\mathfrak{Z}}_n^m=\emptyset$ because $\operatorname{im}\mathfrak{f}_{l,z}^m=B(z,\mathfrak{r}_l)\subset\mathfrak{B}_n^l(z)$ for all $(l,z)\in\mathfrak{P}_n^m$. Then we set $\mathfrak{Z}_n^m=\widetilde{\mathfrak{Z}}_n^m\sqcup\widehat{\mathfrak{Z}}_n^m$. Let us define the maps $\mathfrak{f}_{n,z}^m$ depending on whether $z\in\widehat{\mathfrak{Z}}_n^m$ or $z\in\widetilde{\mathfrak{Z}}_n^m$. If $z\in\widehat{\mathfrak{Z}}_n^m$, let $\mathfrak{f}_{n,z}^m$ be any pointed isometry from $(B(p,\mathfrak{r}_n),p)$ to $(B(z,\mathfrak{r}_n),z)$, which exists because $z\in\Omega_n$. In the case where $z\in\widetilde{\mathfrak{Z}}_n^m$, using Claim 3.3.3 (b) and since $\operatorname{im}\mathfrak{f}_{l',z'}^m\subset\mathfrak{B}_n^{l'}(z')$ for all $(l,z)\in\overline{\mathfrak{P}}_n^m$, there is a unique $(l',z')\in\overline{\mathfrak{P}}_n^m$ such that $z\in\mathfrak{B}_n^{l'}(z')$, and therefore $z\in\mathfrak{f}_{l',z'}^m(\mathfrak{Z}_n^{l'})$. Let $z''=(\mathfrak{f}_{l',z'}^m)^{-1}(z)$. Then we define $\mathfrak{f}_{n,z}^m=\mathfrak{f}_{l',z'}^m\circ\mathfrak{f}_{n,z''}^{l'}$. We proceed hereafter to prove that these definitions satisfy all required properties.

In order to prove (i), we shall first establish the following claims.

Claim 3.3.4. We have $\mathfrak{Z}_n^m\subset\Omega_n\cap\mathfrak{V}_n^m(p)\setminus\mathfrak{Q}_n^m$.

Using the induction hypothesis with (i), we know that for each $l\in\mathbb{N}$ with $n<l<m$, we have

$$\mathfrak{Z}_n^l\subset\Omega_n\cap\mathfrak{V}_n^l(p)\setminus\mathfrak{Q}_n^l. \tag{3.34}$$

Let us prove that

$$\widetilde{\mathfrak{Z}}_n^m\subset\left(\bigsqcup_{(l,z)\in\overline{\mathfrak{P}}_n^m}\mathfrak{B}_n^l(z)\right)\cap\Omega_n\setminus\mathfrak{Q}_n^m. \tag{3.35}$$

Let $(l,z) \in \overline{\mathfrak{P}}^m_n$. By the induction hypothesis, $\mathfrak{f}^m_{l,z} \colon (B(p,\mathfrak{r}_l),p) \to (B(z,\mathfrak{r}_l),z)$ is a pointed isometry, and therefore $\mathfrak{f}^m_{l,z}(\mathfrak{V}^l_n(p)) = \mathfrak{V}^l_n(z)$. Using now 3.30, we obtain

$$\mathfrak{B}^l_n(z) \setminus \mathfrak{Q}^m_n = \mathfrak{V}^l_n(z) \setminus \mathfrak{f}^m_{l,z}\left(\mathfrak{Q}^l_n\right) = \mathfrak{f}^m_{l,z}\left(\mathfrak{V}^l_n(p)\right) \setminus \mathfrak{f}^m_{l,z}\left(\mathfrak{Q}^l_n\right) = \mathfrak{f}^m_{l,z}\left(\mathfrak{V}^l_n(p) \setminus \mathfrak{Q}^l_n\right),$$

and by (3.34), we conclude that $\mathfrak{f}^m_{l,z}(\mathfrak{Z}^l_n) \subset \mathfrak{B}^l_n(z) \setminus \mathfrak{Q}^m_n$. Let now $z' \in \mathfrak{Z}^l_n$. By the induction hypothesis with (i), we have $z' \in \Omega_n \cap \mathfrak{V}^l_n(p)$. By (3.25), we have that $d(p,z') \le \mathfrak{r}_l - \mathfrak{t}_n$. So, by the triangle inequality we obtain $B(z',\mathfrak{r}_n) \subset B(p, \mathfrak{r}_l - \mathfrak{t}_n + \mathfrak{r}_n)$. Applying (3.24), we get $B(z',\mathfrak{r}_n) \subset B(p,\mathfrak{r}_l) = \operatorname{dom}(\mathfrak{f}^m_{l,z})$. Thus, the fact that $\mathfrak{f}^m_{l,z}$ is an isometry implies that $\mathfrak{f}^m_{l,z}(z') \in \Omega_n$, obtaining $\mathfrak{f}^m_{l,z}(\mathfrak{Z}^l_n) \subset \Omega_n$. Therefore, for every $(l,z) \in \overline{\mathfrak{P}}^m_n$, $\mathfrak{f}^m_{l,z}(\mathfrak{Z}^l_n) \subset \mathfrak{B}^l_n(z) \cap \Omega_n \setminus \mathfrak{Q}^m_n$, and we get (3.35).

By definition,

$$\widehat{\mathfrak{Z}}^m_n \subset \Omega_n \cap \mathfrak{V}^m_n(p) \setminus \bigsqcup_{(l,z)\in\overline{\mathfrak{P}}^m_n} \mathfrak{B}^l_n(z).$$

Using now 3.31 and (3.35), we obtain

$$\begin{aligned} \mathfrak{Z}^m_n &= \widetilde{\mathfrak{Z}}^m_n \sqcup \widehat{\mathfrak{Z}}^m_n \\ &\subset \left(\left(\bigsqcup_{(l,z)\in\overline{\mathfrak{P}}^m_n} \mathfrak{B}^l_n(z)\right) \cap \Omega_n \setminus \mathfrak{Q}^m_n\right) \sqcup \left(\Omega_n \cap \mathfrak{V}^m_n(p) \setminus \bigsqcup_{(l,z)\in\overline{\mathfrak{P}}^m_n} \mathfrak{B}^l_n(z)\right) \\ &= \Omega_n \cap \mathfrak{V}^m_n(p) \setminus \mathfrak{Q}^m_n, \end{aligned}$$

completing the proof of Claim 3.3.4.

Claim 3.3.5. (a) For any $(l,z) \in \mathfrak{P}^m_n$, we have $d(\mathfrak{V}^l_n(z), X \setminus \mathfrak{B}^l_n(z)) \ge \mathfrak{s}_n$.

(b) For all $(l,z) \neq (l',z')$ in $\overline{\mathfrak{P}}^m_n$, we have $\mathfrak{B}^l_n(z) \cap \mathfrak{B}^{l'}_n(z') = \emptyset$; in particular, one has $d(\mathfrak{V}^l_n(z), \mathfrak{V}^{l'}_n(z')) \ge \mathfrak{s}_n$.

Let us prove (a). Suppose we have two points $x \in \mathfrak{V}^l_n(z)$, and $x' \in X \setminus \mathfrak{B}^l_n(z)$ such that $d(x,x') < \mathfrak{s}_n$. By the definition of $\mathfrak{V}^l_n(z)$ in (3.25), we have $d(z,x) \le \mathfrak{r}_l - \mathfrak{t}_n$. Using now the triangle inequality, we conclude $d(z,x') \le \mathfrak{r}_l - \mathfrak{t}_n + \mathfrak{s}_n \le \mathfrak{r}_l + \mathfrak{s}_n$. This inequality implies that $x' \in \mathfrak{B}^l_n(z)$, a contradiction.

Let us prove (b). If $l = l'$, the result follows from Claim 3.3.2 (a). Suppose then that $l < l'$. We have $\mathfrak{B}^l_n(z) \cap \mathfrak{B}^{l'}_n(z') = \emptyset$, otherwise $(l,z) < (l',z')$ by Claim 3.3.3 (a), contradicting the maximality of (l,z). Then, by (3.25), we have

$$d(z,z') \ge \mathfrak{r}_l + \mathfrak{r}_{l'} + 2\mathfrak{s}_n > \mathfrak{r}_l + \mathfrak{r}_{l'} - 2\mathfrak{t}_n + \mathfrak{s}_n.$$

Then (b) follows from the definition of $\mathfrak{V}^l_n(z)$ in (3.25). This completes the proof of Claim 3.3.5.

Claim 3.3.6. The set $\mathfrak{Z}^m_n$ is $\mathfrak{s}_n$-separated.

We will first show that $\widetilde{\mathfrak{Z}}_n^m$ is $\mathfrak{s}_n$-separated. Let $(l,z) \in \overline{\mathfrak{P}}_n^m$. By the induction hypothesis, the set $\mathfrak{Z}_n^l$ is $\mathfrak{s}_n$-separated, and the map $\mathfrak{f}_{l,z}^m$ is an isometry. So the set $\mathfrak{f}_{l,z}^m(\mathfrak{Z}_n^l)$ is also $\mathfrak{s}_n$-separated. From Claim 3.3.5 (b), the induction hypothesis with (i) and the definition of $\widetilde{\mathfrak{Z}}_n^m$, it is immediate that $\widetilde{\mathfrak{Z}}_n^m$ is also $\mathfrak{s}_n$-separated. Recall that the set $\widehat{\mathfrak{Z}}_n^m$ is $\mathfrak{s}_n$-separated by construction. Therefore, to prove that $\mathfrak{Z}_n^m = \widehat{\mathfrak{Z}}_n^m \sqcup \widetilde{\mathfrak{Z}}_n^m$ is $\mathfrak{s}_n$-separated, it suffices to show that $d(\widetilde{\mathfrak{Z}}_n^m, \widehat{\mathfrak{Z}}_n^m) \geq \mathfrak{s}_n$. Let $z \in \widetilde{\mathfrak{Z}}_n^m$ and $z' \in \widehat{\mathfrak{Z}}_n^m$. By construction and the induction hypothesis with (i), there is some $(l, z'') \in \overline{\mathfrak{P}}_n^m$ such that $z \in \mathfrak{V}_n^l(z'')$ and $z' \notin \mathfrak{B}_n^l(z'')$, and Claim 3.3.6 then follows by Claim 3.3.5 (a).

Let us prove (i). By Claims 3.3.4 and 3.3.6, we have that $\mathfrak{Z}_n^m$ is an $\mathfrak{s}_n$-separated subset of $\Omega_n \cap \mathfrak{V}_n^m(p) \setminus \mathfrak{Q}_n^m$. Therefore, we only need to establish maximality among the $\mathfrak{s}_n$-separated subsets of $\Omega_n \cap \mathfrak{V}_n^m(p) \setminus \mathfrak{Q}_n^m$. By the induction hypothesis with (i), for any $(l,z) \in \overline{\mathfrak{P}}_n^m$, the set $\mathfrak{Z}_n^l$ is a maximal $\mathfrak{s}_n$-separated subset of $\Omega_n \cap \mathfrak{V}_n^l(p) \setminus \mathfrak{Q}_n^l$. This last set is contained in $B(p, \mathfrak{r}_l) = \operatorname{dom}(\mathfrak{f}_{l,z}^m)$, and the map $\mathfrak{f}_{l,z}^m \colon B(p,\mathfrak{r}_l) \to B(z, \mathfrak{r}_l)$ is an isometry. Thus, for any $(l,z) \in \overline{\mathfrak{P}}_n^m$, the set $\mathfrak{f}_{l,z}^m(\mathfrak{Z}_n^l)$ is a maximal $\mathfrak{s}_n$-separated subset of

$$\mathfrak{f}_{l,z}^m \left(\Omega_n \cap \mathfrak{V}_n^l(p) \setminus \mathfrak{Q}_n^l\right) = \Omega_n \cap \mathfrak{V}_n^l(z) \setminus \mathfrak{Q}_n^m,$$

where the last equality holds by 3.30. So, by Claim 3.3.5 (b), the set $\widetilde{\mathfrak{Z}}_n^m$ is a maximal $\mathfrak{s}_n$-separated subset of

$$\Omega_n \cap \bigsqcup_{(l,z) \in \overline{\mathfrak{P}}_n^m} \mathfrak{V}_n^l(z) \setminus \mathfrak{Q}_n^m.$$

In turn, by construction, the set $\widehat{\mathfrak{Z}}_n^m$ was a maximal $\mathfrak{s}_n$-separated subset of

$$\Omega_n \cap \mathfrak{V}_n^m(p) \setminus \bigsqcup_{(l,z) \in \overline{\mathfrak{P}}_n^m} \mathfrak{B}_n^l(z).$$

Therefore, using 3.31 and Claim 3.3.5 (a), we get that $\mathfrak{Z}_n^m$ is a maximal $\mathfrak{s}_n$-separated subset of $\Omega_n \cap \mathfrak{V}_n^m \setminus \mathfrak{Q}_n^m$.

Let us prove (ii). Let $x \in \mathfrak{Z}_n^m$, and $(l,z) \in \mathfrak{P}_n^m$. By property 3.3.2 (i), it holds that $x \in X \setminus \mathfrak{C}_n^l(z)$, so we conclude that either $x \in \mathfrak{V}_n^l(z)$ or $x \notin \mathfrak{B}_n^l(z)$. In the former case (b) holds, as can be seen using Lemma 3.3.1 (iii); and in the latter case (a) follows from Lemma 3.3.1 (ii).

Let us prove (iii). Let $(l,z) \in \mathfrak{P}_n^m$. We have defined $\widehat{\mathfrak{Z}}_n^m$ so that

$$\widehat{\mathfrak{Z}}_n^m \subset \mathfrak{V}_n^m(p) \setminus \bigsqcup_{(l',z') \in \overline{\mathfrak{P}}_n^m} \mathfrak{B}_n^{l'}(z').$$

Therefore $\widehat{\mathfrak{Z}}_n^m \cap \mathfrak{B}_n^l(z) = \emptyset$. It is then clear that it only remains to show that $\widetilde{\mathfrak{Z}}_n^m \cap \mathfrak{B}_n^l(z) = \mathfrak{f}_{l,z}^m(\mathfrak{Z}_n^l)$. We will consider first the case where $(l,z) \in \overline{\mathfrak{P}}_n^m$. It follows from Claim 3.3.2 (a) and (i) that, for any $(l',z') \in \overline{\mathfrak{P}}_n^m$ such that $(l',z') \neq (l,z)$, we have

$$\mathfrak{B}_n^l(z) \cap \mathfrak{f}_{l',z'}^m\left(\mathfrak{Z}_n^{l'}\right) \subset \mathfrak{B}_n^l(z) \cap \mathfrak{B}_n^{l'}(z') = \emptyset.$$

Additionally, it is obvious that $\mathfrak{f}^m_{l,z}(\mathfrak{Z}^l_n) \cap \mathfrak{B}^l_n(z) = \mathfrak{f}^m_{l,z}(\mathfrak{Z}^l_n)$, and therefore $\widetilde{\mathfrak{Z}}^m_n \cap \mathfrak{B}^l_n(z) = \mathfrak{f}^m_{l,z}(\mathfrak{Z}^l_n)$.

Suppose now that $(l,z) \in \mathfrak{P}^m_n \setminus \overline{\mathfrak{P}}^m_n$. Then, according to Claim 3.3.3 (b), there is a unique $(l',z)' \in \overline{\mathfrak{P}}^m_n$ such that $(l,z) < (l',z')$. We have already proved that $\mathfrak{Z}^m_n \cap \mathfrak{B}^{l'}_n(z') = \mathfrak{f}^m_{l',z'}(\mathfrak{Z}^{l'}_n)$. Let $y = (\mathfrak{f}^m_{l',z'})^{-1}(z)$. By the induction hypothesis with (iii), we know that $\mathfrak{Z}^{l'}_n \cap \mathfrak{B}^l_n(y) = \mathfrak{f}^{l'}_{l,y}(\mathfrak{Z}^l_n)$. The fact that $\mathfrak{f}^m_{l',z'}$ is an isometry and $\mathfrak{B}^l_n(z) \subset B(z', \mathfrak{r}_{l'})$ implies that

$$\mathfrak{Z}^m_n \cap \mathfrak{B}^l_n(z) = \mathfrak{f}^m_{l',z'}\left(\mathfrak{Z}^{l'}_n\right) \cap \mathfrak{B}^l_n(z) = \mathfrak{f}^m_{l',z'}\left(\mathfrak{Z}^{l'}_n \cap \mathfrak{B}^l_n(y)\right) = \mathfrak{f}^m_{l',z'}\left(\mathfrak{f}^m_{l,y}\left(\mathfrak{Z}^l_n\right)\right).$$

Then, using the induction hypothesis with (iv), we know that $\mathfrak{f}^m_{l,z} = \mathfrak{f}^m_{l',z'} \circ \mathfrak{f}^{l'}_{l,y}$, obtaining $\mathfrak{Z}^m_n \cap \mathfrak{B}^l_n(z) = \mathfrak{f}^m_{l,z}(\mathfrak{Z}^l_n)$.

Let us prove (iv). As it was shown in the proof of (iii), if $x \in \mathfrak{Z}^m_n$ and $(l,z) \in \mathfrak{P}^m_n$ are such that (b) holds, then $x \in \widetilde{\mathfrak{Z}}^m_n$. Consider first the case where $(l,z) \in \overline{\mathfrak{P}}^m_n$. Then the equality $\mathfrak{f}^m_{n,x} = \mathfrak{f}^m_{l,z} \circ \mathfrak{f}^l_{n,x'}$ for $x' = (\mathfrak{f}^m_{l,z})^{-1}(x)$ is precisely the definition of $\mathfrak{f}^m_{n,z}$. Therefore we can suppose that $(l,z) \in \mathfrak{P}^m_n \setminus \overline{\mathfrak{P}}^m_n$. According to Claim 3.3.3 (b) we know that there is a unique $(l',z') \in \overline{\mathfrak{P}}^m_n$ such that $(l,z) < (l',z')$, and x satisfies (b) also with (l',z'). We have already proved that, for $x' = (\mathfrak{f}^m_{l',z'})^{-1}(x)$, the equality $\mathfrak{f}^m_{n,x} = \mathfrak{f}^m_{l',z'} \circ \mathfrak{f}^{l'}_{n,x'}$ holds. Moreover, by the induction hypothesis with (iv), if $y = (\mathfrak{f}^m_{l',z'})^{-1}(z)$ and $x'' = (\mathfrak{f}^m_{l,z})^{-1}(x)$, we have $(\mathfrak{f}^{l'}_{l,y})^{-1}(x') = x''$ and

$$\mathfrak{f}^m_{l,z} = \mathfrak{f}^m_{l',z'} \circ \mathfrak{f}^{l'}_{l,y}, \quad \mathfrak{f}^{l'}_{n,x'} = \mathfrak{f}^{l'}_{l,y} \circ \mathfrak{f}^l_{n,x''}.$$

Therefore

$$\mathfrak{f}^m_{n,x} = \mathfrak{f}^m_{l',z'} \circ \mathfrak{f}^{l'}_{n,x'} = \mathfrak{f}^m_{l',z'} \circ \mathfrak{f}^{l'}_{l,y} \circ \mathfrak{f}^l_{n,x''} = \mathfrak{f}^m_{l,z} \circ \mathfrak{f}^l_{n,x''},$$

as desired.

To prove (v), we need the following:

Claim 3.3.7. $\mathfrak{Z}^{m-1}_n \subset \mathfrak{Z}^m_n$, and, for all $z \in \mathfrak{Z}^{m-1}_n$, we have $\mathfrak{f}^m_{n,z} = \mathfrak{f}^{m-1}_{n,z}$.

Let $z \in \mathfrak{Z}^{m-1}_n$. By (i) and the induction hypothesis with (vi), we have $z \in \mathfrak{V}^{m-1}_n(p) \subset B(p, \mathfrak{r}_{m-1})$, $p \in \mathfrak{Z}^m_{m-1}$ and $\mathfrak{f}^m_{m-1,p} = \mathrm{id}_{B(p,\mathfrak{r}_{m-1})}$. By the definition of $\mathfrak{P}^m_n$ in (3.26), it is immediate that $(m-1,p) \subset \overline{\mathfrak{P}}^m_n$. Then $z \in \mathfrak{Z}^m_{m-1} = \mathfrak{f}^m_{m-1,p}(\mathfrak{Z}^m_{m-1}) \subset \widetilde{\mathfrak{Z}}^m_n$. Using (iv), we see that $\mathfrak{f}^m_{n,z} = \mathfrak{f}^m_{m-1,p} \circ \mathfrak{f}^{m-1}_{n,z} = \mathrm{id}_{B(p,\mathfrak{r}_{m-1})} \circ \mathfrak{f}^{m-1}_{n,z} = \mathfrak{f}^{m-1}_{n,z}$.

Claim 3.3.8. $\mathfrak{Z}^m_{n+1} \subset \mathfrak{Z}^m_n$, and, for each $z \in \mathfrak{Z}^m_{n+1}$, we have $\mathfrak{f}^m_{n,z} = \mathfrak{f}^m_{n+1,z}|_{B(p,\mathfrak{r}_n)}$.

Let $z \in \mathfrak{Z}^m_{n+1}$. Then clearly $(n+1,z) \in \mathfrak{P}^m_n$. The map $\mathfrak{f}^m_{n+1,z} \colon (B(p,\mathfrak{r}_{n+1}),p) \to (B(z,\mathfrak{r}_{n+1}),z)$ is by definition a pointed isometry. So, $(\mathfrak{f}^m_{n+1,z})^{-1}(z) = p$, and, by (iv) we obtain $\mathfrak{f}^m_{n,z} = \mathfrak{f}^m_{n+1,z} \circ \mathfrak{f}^{n+1}_{n,p}$. We have already proved that $\mathfrak{f}^{n+1}_{n,p} = \mathrm{id}_{B(p,\mathfrak{r}_n)}$. So $\mathfrak{f}^m_{n,z} = \mathfrak{f}^m_{n+1,z} \circ \mathrm{id}_{B(p,\mathfrak{r}_n)} = \mathfrak{f}^m_{n+1,z}|_{B(p,\mathfrak{r}_n)}$, completing the proof of the claim.

Then (v) follows from Claims 3.3.7 and 3.3.8 by induction.

Property (vi) follows easily from (v) and the definition of $\mathfrak{Z}_n^n$ and $\mathfrak{f}_{n,p}^n$. This concludes the proof of Proposition 3.3.2. □

For $n < m$, let $\mathfrak{c}_n^m \colon B(p, \mathfrak{r}_m) \to \{n+1, \dots, m\}$ be defined by

$$\mathfrak{c}_n^m(x) = \min\{\, n \in \mathbb{Z} \mid n < l \leq m,\ \exists z \in \mathfrak{Z}_l^m,\ x \in B(z, \mathfrak{r}_l)\,\}$$

Since the set $\mathfrak{Z}_l^m$ is $2\mathfrak{r}_l$-separated by Proposition 3.3.2 (i) and (3.23), if $x \in B(z, \mathfrak{r}_l)$ for some $z \in \mathfrak{Z}_l^m$, then z is the unique point in $\mathfrak{Z}_l^m$ that satisfies this condition. Let $\mathfrak{p}_n^m \colon \mathfrak{Z}_n^m \to X$ be defined by assigning to every $x \in \mathfrak{Z}_n^m$ the unique point $\mathfrak{p}_n^m(x)$ in $\mathfrak{Z}_{\mathfrak{c}_n^m(x)}^m$ satisfying $x \in B(\mathfrak{p}_n^m(x), \mathfrak{r}_{\mathfrak{c}_n^m(x)})$.

For $n \in \mathbb{N}$, let $\preceq_n^n$ be the trivial order relation on $\mathfrak{Z}_n^n = \{p\}$.

Proposition 3.3.3. *For $0 \leq n < m$, there is an order*[1] *relation $\preceq_n^m$ on $\mathfrak{Z}_n^m$ such that:*

(i) *p is the least element of $(\mathfrak{Z}_n^m, \preceq_n^m)$;*

(ii) *for $x, y \in \mathfrak{Z}_n^m$, if $\mathfrak{c}_n^m(x) < \mathfrak{c}_n^m(y)$, then $x \prec_n^m y$ (meaning $x \preceq_n^m y$ and $x \neq y$); and,*

(iii) *for any $(l, z) \in \mathfrak{P}_n^m$, the map $\mathfrak{f}_{l,z}^m \colon (\mathfrak{Z}_n^l, \preceq_n^l) \to (\mathfrak{Z}_n^m \cap B(z, \mathfrak{r}_l), \preceq_n^m)$ is order preserving.*

Proof. We proceed by induction like in Proposition 3.3.2. Let $\preceq_n^{n+1}$ be an arbitrary ordering of $\mathfrak{Z}_n^{n+1}$ whose least element is p. For $m = n+1$, we have $\mathfrak{c}_n^m(x) = m$ for every $x \in \mathfrak{Z}_n^m$ if $\mathfrak{P}_n^m = \emptyset$. Thus (ii) and (iii) are trivially satisfied in this case.

Suppose now that we have defined $\preceq_k^l$ when either $l > n$, or $l = n$ and $k < m$. Let $\trianglelefteq_n^m$ be an arbitrary ordering of $B(p, \mathfrak{r}_m) \setminus \bigcup_{(l,z) \in \mathfrak{P}_n^m} B(z, \mathfrak{r}_l)$. Then we define $\preceq_n^m$ using several cases as follows:

(a) if $\mathfrak{c}_n^m(x) < \mathfrak{c}_n^m(y)$, then $x \prec_n^m y$;

(b) if $\mathfrak{c}_n^m(x) = \mathfrak{c}_n^m(y) < m$ and $\mathfrak{p}_n^m(x) = \mathfrak{p}_n^m(y)$, then $x \preceq_n^m y$ if and only if

$$\left(\mathfrak{f}_{\mathfrak{c}_n^m(x), \mathfrak{p}_n^m(x)}^m\right)^{-1}(x) \preceq_n^{\mathfrak{c}_n^m(x)} \left(\mathfrak{f}_{\mathfrak{c}_n^m(x), \mathfrak{p}_n^m(x)}^m\right)^{-1}(y);$$

(c) if $\mathfrak{c}_n^m(x) = \mathfrak{c}_n^m(y) < m$ and $\mathfrak{p}_n^m(x) \neq \mathfrak{p}_n^m(y)$, then $x \prec_n^m y$ if and only if $\mathfrak{p}_n^m(x) \prec_{\mathfrak{c}_n^m(x)}^m \mathfrak{p}_n^m(y)$; and,

(d) if $\mathfrak{c}_n^m(x) = \mathfrak{c}_n^m(y) = m$, then $x \preceq_n^m y$ if and only if $x \trianglelefteq_n^m y$.

[1]In the order relations, it is assumed that any pair of elements is comparable. When this property is not satisfied, we use the term partial order relation.

It can be easily checked that this is indeed an order relation, and it is obvious that it satisfies (i) and (ii). Let us prove that it also satisfies (iii). Suppose first that $(l,z) \in \overline{\mathfrak{P}}_n^m$. For any $x, y \in B(z, \mathfrak{r}_l)$, clearly, $\mathfrak{c}_n^m(x) = \mathfrak{c}_n^m(y) = l$ and $\mathfrak{p}_n^m(x) = \mathfrak{p}_n^m(y) = z$, and therefore $\mathfrak{f}_{l,z}^m$ is order preserving by (b).

Suppose now that $(l,z) \in \mathfrak{P}_n^m \setminus \overline{\mathfrak{P}}_n^m$, and let $(l',z') \in \overline{\mathfrak{P}}_n^m$ be the unique maximal element such that $(l,z) < (l',z')$, given by Claim 3.3.3 (b). Let $z'' = (\mathfrak{f}_{l',z'}^m)^{-1}(z)$. By the induction hypothesis, the map

$$\mathfrak{f}_{l,z''}^m \colon \left(\mathfrak{Z}_n^l, \preceq_n^l\right) \to \left(\mathfrak{Z}_n^{l'} \cap B(z'', \mathfrak{r}_l), \preceq_n^{l'}\right)$$

is order preserving, and

$$\mathfrak{f}_{l',z'}^m \colon (\mathfrak{Z}_n^{l'}, \preceq_n^{l'}) \to (\mathfrak{Z}_n^m \cap B(z', \mathfrak{r}_{l'}), \preceq_n^m)$$

is order preserving because $(l',z') \in \overline{\mathfrak{P}}_n^m$. Therefore

$$\mathfrak{f}_{l,z}^m = \mathfrak{f}_{l',z'}^m \circ \mathfrak{f}_{l,z''}^m \colon \left(\mathfrak{Z}_n^l, \preceq_n^l\right) \to (\mathfrak{Z}_n^m \cap B(z, \mathfrak{r}_l), \preceq_n^m)$$

is also order preserving. $\square$

Define

$$\mathfrak{X}_n = \bigcup_{m \geq n} \mathfrak{Z}_n^m,$$

$$\mathfrak{R}_n = \bigcup_{m \geq n} \mathfrak{P}_n^m = \{\, (m,x) \in \mathbb{N} \times X \mid n < m,\ x \in \mathfrak{X}_m \,\}, \tag{3.36}$$

$$\mathfrak{S}_n = \bigcup_{(m,x) \in \mathfrak{R}_n} \mathfrak{C}_n^m(x) = \bigcup_{m \geq n} \mathfrak{Q}_n^m. \tag{3.37}$$

For $n \in \mathbb{N}$ and $x \in \mathfrak{X}_n$, there is some $m \geq n$ such that $x \in \mathfrak{Z}_n^m$. Thus, define $\mathfrak{h}_{n,x} = \mathfrak{f}_{n,x}^m \colon (B(p, \mathfrak{r}_n), p) \to (B(x, \mathfrak{r}_n), x)$, where Proposition 3.3.2 (v) ensures that this does not depend on m.

Let $<$ be the binary relation on $\mathfrak{R}_n$ defined by declaring $(m,x) < (m',x')$ if $m < m'$ and $B(x, \mathfrak{r}_m) \subset B(x', \mathfrak{r}_{m'})$, and let $\leq$ be the reflexive closure of $<$.

The next proposition follows immediately from the definitions.

Proposition 3.3.4. *For $n \in \mathbb{N}$, we have the following properties:*

(i) *The set $\mathfrak{X}_n$ is a maximal $\mathfrak{s}_n$-separated subset of $(X \setminus \mathfrak{S}_n, d)$ containing p.*

(ii) *For any $x \in \mathfrak{X}_n$ and $(m,y) \in \mathfrak{R}_n$,*

  (a) *either $x \notin \mathfrak{B}_n^m(y)$ and, for $0 \leq l < n$, we have $\mathfrak{B}_l^n(x) \cap \mathfrak{B}_l^m(y) = \emptyset$, or*

(b) $x \in \mathfrak{V}_n^m(y)$ *and, for* $0 \le l < n$, *we have* $\mathfrak{B}_l^n(x) \subset \mathfrak{V}_l^m(y)$.

(iii) *For any* $(m,x) \in \mathfrak{R}_n$, *we have* $\mathfrak{X}_n \cap \mathfrak{B}_n^m(x) = \mathfrak{h}_{m,x}(\mathfrak{Z}_n^m)$.

(iv) *For any* $x \in \mathfrak{X}_n$ *and* $(m,y) \in \mathfrak{R}_n$ *such that* $B(x,\mathfrak{r}_n) \subset B(y,\mathfrak{r}_m)$, *we have* $\mathfrak{h}_{n,x} = \mathfrak{h}_{m,y} \circ \mathfrak{h}_{n,x'}$ *with* $x' = \mathfrak{h}_{m,y}^{-1}(x)$.

(v) *For* $n \le m$, *we have* $\mathfrak{X}_m \subset \mathfrak{X}_n$, *and, for* $x \in \mathfrak{X}_m$, $\mathfrak{h}_{n,x} = \mathfrak{h}_{m,x}|_{B(p,\mathfrak{r}_n)}$.

(vi) *We have* $p \in \mathfrak{X}_n$ *and* $\mathfrak{h}_{n,p} = \mathrm{id}_{B(p,\mathfrak{r}_n)}$.

Proof. Let us prove (i). By 3.30, Proposition 3.3.2 (vi) and (3.37), for every $m' > m > n$, we have

$$\mathfrak{V}_n^m(p) \setminus \mathfrak{Q}_n^m = \mathfrak{V}_n^m(p) \setminus \mathfrak{f}_{m,p}^{m'}(\mathfrak{Q}_n^m) = \mathfrak{V}_n^m(p) \setminus \big(\mathfrak{Q}_n^{m'} \cap \mathfrak{B}_n^m(p)\big) = \mathfrak{V}_n^m(p) \setminus \mathfrak{Q}_n^{m'} = \mathfrak{V}_n^m(p) \setminus \mathfrak{S}_n.$$

Hence the sets $\Omega_n \cap \mathfrak{V}_n^m(p) \setminus \mathfrak{Q}_n^m$, for $m > n$, form an increasing chain whose union is $\Omega_n \cap X \setminus \mathfrak{S}_n$. By Proposition 3.3.2 (i),(v), it follows that $\mathfrak{X}_n$ is a maximal $\mathfrak{s}_n$-separated subset of $\Omega_n \cap X \setminus \mathfrak{S}_n$, being the union of an increasing sequence of maximal $\mathfrak{s}_n$-separated subsets of the increasing sequence of sets $\Omega_n \cap \mathfrak{V}_n^m(p) \setminus \mathfrak{Q}_n^m$. Finally, $p \in \mathfrak{X}_n$ follows from Proposition 3.3.2 (vi). This completes the proof of (i).

The remaning properties are direct consequences of the corresponding properties of Proposition 3.3.2, using that the sets $\mathfrak{Z}_n^m$, for $m > n$, form an increasing chain (Proposition 3.3.2 (v)). $\square$

By Propositions 3.3.2 (vi) and 3.3.3 (iii), the order relations $\preceq_n^m$, $m \ge n$, define an order relation $\preceq_n$ on $\mathfrak{X}_n$. The following proposition is a consequence of Proposition 3.3.3.

Proposition 3.3.5. *For* $n \in \mathbb{N}$, *the following holds:*

(i) *The point* p *is the least element.*

(ii) *For* $x, y \in \mathfrak{X}_n$, *if* $\mathfrak{c}_n(x) < \mathfrak{c}_n(y)$, *then* $x \prec_n y$.

(iii) *For any* $(l,z) \in \mathfrak{R}_n$, *the map* $\mathfrak{h}_{l,z} \colon (\mathfrak{Z}_n^l, \preceq_n^l) \to (\mathfrak{X}_n \cap B(z,\mathfrak{r}_l), \preceq_n)$ *is order preserving.*

Lemma 3.3.6. *For any* $(m,x) \in \mathfrak{R}_n$, *we have* $C(x, \mathfrak{r}_m - \mathfrak{t}_n - 2\omega_n - 1, \mathfrak{r}_m - \mathfrak{t}_n) \cap \mathfrak{S}_n = \emptyset$.

Proof. First, note that $\mathfrak{r}_m - \mathfrak{t}_n - 2\omega_n - 1 > 0$ by (3.22). For any $(m,x) \in \mathfrak{R}_n$, the following inclusion holds according to (3.25):

$$C(x, \mathfrak{r}_m - \mathfrak{t}_n - 2\omega_n - 1, \mathfrak{r}_m - \mathfrak{t}_n) \subset \mathfrak{V}_n^m(x).$$

So, by Proposition 3.3.4 (ii), if $(l, z) \in \mathfrak{R}_n$ satisfies

$$C(x, \mathfrak{r}_m - \mathfrak{t}_n - 2\omega_n - 1, \mathfrak{r}_m - \mathfrak{t}_n) \cap \mathfrak{C}_n^l(z) \neq \emptyset, \tag{3.38}$$

then $z \in \mathfrak{V}_l^m(x)$ and $\mathfrak{B}_n^l(z) \subset \mathfrak{V}_n^m(x)$; in particular, $(l, z) < (m, x)$. Since, by (3.24),

$$\mathfrak{r}_m - \mathfrak{t}_l + \mathfrak{r}_l + \mathfrak{s}_n \leq \mathfrak{r}_m - \mathfrak{t}_n - 2\omega_n - 1,$$

it follows that

$$\mathfrak{C}_n^l(z) \subset \mathfrak{B}_n^l(z) \subset \mathfrak{V}_n^m(x) \subset B(x, \mathfrak{r}_m - \mathfrak{t}_l + \mathfrak{r}_l + \mathfrak{s}_n) \subset B(x, \mathfrak{r}_m - \mathfrak{t}_n - 2\omega_n - 1),$$

contradicting (3.38). This shows the statement according to (3.37). □

Proposition 3.3.7. *The set $\Omega_n \setminus \mathfrak{S}_n$ is a $(\mathfrak{s}_n + \mathfrak{t}_n + 3\omega_n)$-net in X.*

Proof. For $u \in X$, let us prove that there is some $x \in \Omega_n \setminus \mathfrak{S}_n$ such that $d(u, x) \leq \mathfrak{s}_n + \mathfrak{t}_n + 3\omega_n$. Since Ω_n is an ω_n-net in X, there is some point $x' \in \Omega_n$ such that $d(u, x') \leq \omega_n$. If $x' \notin \mathfrak{S}_n$, then the desired inequality holds with $x = x'$. Thus, according to (3.37), suppose that there is some $(m, y) \in \mathfrak{R}_n$ such that $x' \in \mathfrak{C}_n^m(y)$. Then $\mathfrak{r}_m - \mathfrak{t}_n < \delta := d(x', y) \leq \mathfrak{r}_m + \mathfrak{s}_n$ according to (3.25). Let $\tau \colon \{0, \ldots, \delta\} \to X$ be a geodesic such that $\tau(0) = y$ and $\tau(\delta) = x'$. Let $v = \tau(\mathfrak{r}_m - \mathfrak{t}_n - \omega_n)$. We have

$$\begin{aligned} d(x', v) &\leq \mathfrak{s}_n + \mathfrak{t}_n + \omega_n, \\ B(v, \omega_n) &\subset C(y, \mathfrak{r}_m - \mathfrak{t}_n - 2\omega_n - 1, \mathfrak{r}_m - \mathfrak{t}_n). \end{aligned} \tag{3.39}$$

Again, because Ω_n is an ω_n-net, there is some $x \in \Omega_n \cap B(v, \omega_n)$, and we have $x \in \Omega_n \setminus \mathfrak{S}_n$ by Lemma 3.3.6. Then, by the triangle inequality,

$$d(u, x) \leq d(u, x') + d(x', v) + d(v, x) \leq \omega_n + \mathfrak{s}_n + \mathfrak{t}_n + \omega_n + \omega_n = \mathfrak{s}_n + \mathfrak{t}_n + 3\omega_n. \quad \square$$

Corollary 3.3.8. *The set $\mathfrak{X}_n$ is a $(2\mathfrak{s}_n + \mathfrak{t}_n + 3\omega_n)$-net in X.*

Proof. Let $u \in X$. By Proposition 3.3.7, there is some $x' \in \Omega_n \setminus \mathfrak{S}_n$ such that $d(u, x') \leq \mathfrak{s}_n + \mathfrak{t}_n + 3\omega_n$. Suppose that $\mathfrak{X}_n \cap B(x', \mathfrak{s}_n) = \emptyset$. Then $\mathfrak{X}_n \sqcup \{x'\}$ would be an $\mathfrak{s}_n$-separated subset of $\Omega_n \setminus \mathfrak{S}_n$ that strictly contains $\mathfrak{X}_n$, contradicting Proposition 3.3.4 (i). Therefore there must be some $x \in \mathfrak{X}_n \cap B(x', \mathfrak{s}_n)$, and, by the triangle inequality, $d(u, x) \leq 2\mathfrak{s}_n + \mathfrak{t}_n + 3\omega_n$. □

For $m \in \mathbb{N}$, let

$$\begin{aligned} \mathfrak{P}_{-1}^m &= \{\, (l, z) \in \mathbb{N} \times X \mid 0 \leq l < m,\ z \in \mathfrak{Z}_l^m \,\}, \\ \mathfrak{R}_{-1} &= \{\, (m, x) \in \mathbb{N} \times X \mid x \in \mathfrak{X}_m \,\}. \end{aligned} \tag{3.40}$$

We can define on both of these sets the relation $(l, z) < (l, z')$ if and only if $l < l'$ and $B(z, \mathfrak{r}_l) \subset B(z', \mathfrak{r}_{l'})$. The induced relations $\leq$ are partial orders and satisfy Claim 3.3.3 (b).

3.4 Construction of X_n

In this section, we show a proposition that will be crucial in the proofs of Theorems 3.1.1 and 3.1.2. Actually, it will be used to prove Theorem 3.1.2, but the same proof applies to Theorem 3.1.1 using a simpler version of the proposition, taking the sets $\mathfrak{X}_n = \emptyset$, and therefore omitting the use of the sets $\mathfrak{R}_n$, numbers $\mathfrak{r}_n$, and maps $\mathfrak{f}^m_{n,z}$ and $\mathfrak{h}_{n,x}$. Thus suppose that X satisfies the hypothesis of Theorem 3.1.2, and consider the notation of Sections 3.2 and 3.3.

For notational convenience, let

$$(X_{-1}, E_{-1}) = (X, E) \;, \quad d_{-1} = d \;, \quad r_{-1} = s_{-1} = R^{\pm}_{-1} = 0 \;, \quad \lambda_{-1} = \Lambda_{-1} = 1. \tag{3.41}$$

For $n \in \mathbb{N}$, we will continue defining constants r_n, subsets $X_n \subset X$ containing $\mathfrak{X}_n$, and a connected graph structure E_n on every X_n with induced metrics d_n. Also, for $x \in X_n$ and $l \in \mathbb{N}$, let $B_n(x, l)$ and $S_n(x, l)$ denote the balls and spheres of center x and radius l in X_n with respect to d_n (recall that, in connected graphs, we use balls defined with non-strict inequalities). With this notation, let $\eta_n : \mathbb{N} \to \mathbb{Q}$ be defined by

$$\eta_n(a) = \exp_2\left(\left\lfloor \frac{a - (\deg X_{n-1})^{10}}{(\deg X_{n-1})^3} \right\rfloor\right). \tag{3.42}$$

Suppose that, for $n \in \mathbb{N}$, the graphs (X_m, E_m) and constants r_m have been defined for integers $-1 \le m < n$. Then let r_n be defined as follows:

(A) If there is some $x \in B_{n-1}(p, \hat{r}_n(2s_n + 1))$ such that

$$(|B_{n-1}(x, \hat{r}_n s_n)| + 6)^2 \ge \eta_n(|B_{n-1}(x, \hat{r}_n)|),$$

then let $r_n = \bar{r}_n$ (see (3.14)).

(B) Otherwise, let $r_n = \hat{r}_n$ (see (3.5) and (3.13)).

Observe that

$$r_0 > 2^{11} \tag{3.43}$$

by (3.6), (3.7), (A) and (B). Moreover, let

$$\Delta_n = \boldsymbol{\Delta}_n(r_0, \dots, r_n) \;, \quad \Lambda_n = \boldsymbol{\Lambda}_n(r_0, \dots, r_n) \;, \quad \Gamma^{\pm}_n = \boldsymbol{\Gamma}^{\pm}_n(r_0, \dots, r_n) \;, \tag{3.44}$$

$$\overline{K}_n = \overline{\boldsymbol{K}}_n(r_0, \dots, r_n) \;,$$

$$K_n = \boldsymbol{K}_n(r_0, \dots, r_n) \;, \quad R^{\pm}_n = \boldsymbol{R}^{\pm}(r_n) \;, \quad \lambda_n = \boldsymbol{\lambda}_n(r_n). \tag{3.45}$$

The functions in (3.44) and (3.45) are all monotone increasing on every coordinate. So, if $\hat{\boldsymbol{r}}_{\boldsymbol{n}}$ denotes the $(n+1)$-tuple $(\hat{r}_0, \dots, \hat{r}_n)$, we get

$$\boldsymbol{\Delta}_n(\hat{\boldsymbol{r}}_n) \le \Delta_n \le \boldsymbol{\Delta}_n(\bar{\boldsymbol{r}}_n), \quad \boldsymbol{R}^{\pm}_n(\hat{r}_n) \le R^{\pm}_n \le \boldsymbol{R}^{\pm}_n(\bar{r}_n), \tag{3.46}$$

and so on. From (3.19), (3.22),(3.44) and (3.45), we get

$$\mathfrak{r}_n > \Gamma_n^\pm \geq R_m^\pm \tag{3.47}$$

for $m = 0, \dots, n$. Finally, let

$$r_n^- = r_n, \quad r_n^+ = r_n s_n. \tag{3.48}$$

By (3.9), (3.16), (3.45) and (3.48), we have

$$r_n^\pm \leq R_n^\pm. \tag{3.49}$$

Proposition 3.4.1. *For $n \in \mathbb{N}$, there are disjoint subsets $X_n^+, X_n^- \subset X$ and a graph structure E_n on $X_n := X_n^- \sqcup X_n^+$ such that the following properties are satisfied:*

(i) $\mathfrak{X}_n \subset X_n \subset X_{n-1}$.

(ii) *For all $(m, x) \in \mathfrak{R}_{n-1}$, we have*

$$\mathfrak{h}_{m,x}\left(X_n^\pm \cap B_{-1}(p, \mathfrak{r}_m - \overline{K}_n)\right) = X_n^\pm \cap B_{-1}(x, \mathfrak{r}_m - \overline{K}_n).$$

(iii) *For all $x \in X_n^\pm$, we have $\eta_n(|B_{n-1}(x, r_n^\pm)|) \geq (6 + |B_{n-1}(x, r_n^\pm s_n)|)^2$.*

(iv) *X_n is a $(2r_n^+ + 1)$-separated R_n^+-net in (X_{n-1}, d_{n-1}).*

(v) *(X_n, E_n) is a connected graph. Let d_n denote the induced metric.*

(vi) *We have $d_n \leq d_{n-1} \leq \lambda_n d_n$ and $d_n \leq d_{-1} \leq \Lambda_n d_n$.*

(vii) *We have $\deg X_n \leq \Delta_n, 4(\deg X_{n-1} - 1)^{2R_n^+}$.*

(viii) *For any $(m, x) \in \mathfrak{R}_{n-1}$, the restriction of $\mathfrak{h}_{m,x}$ to $X_n \cap B_{-1}(p, \mathfrak{r}_m - K_n)$ is an $(s_{n+1}R_{n+1}^+ + \Gamma_n^+)$-short scale isometry with respect to d_n.*

Remark 3.4.1. Note that $K_n < \mathfrak{t}_n, \mathfrak{r}_n$ by (3.22), (3.24) and the fact that $\bar{r}_n > r_n$. This and the inequality $K_n > \overline{K}_n$ yield $\mathfrak{r}_m - \overline{K}_n, \mathfrak{r}_m - K_n > 0$ in (ii) and (viii).

Remark 3.4.2. In accordance with the discussion at the beginning of the section, to prove Theorem 3.1.1, the items (ii) and (viii) must be omitted, and only the inclusion "$X_n \subset X_{n-1}$" must be considered in (i).

The rest of this section is devoted to the proof of the above proposition. We proceed by induction on n. The following lemma follows from Proposition 3.3.4, (3.41) and (3.40). The items are irregularly numbered so that there is an obvious correspondence with those of Proposition 3.4.1.

Lemma 3.4.2. *The following properties hold:*

(i)' $\mathfrak{X}_0 \subset X_{-1}$.

(ii)' *For all* $(m,x) \in \mathfrak{R}_{-1}$, *we have*

$$\mathfrak{h}_{m,x}\left(X_{-1} \cap B_{-1}(p,\mathfrak{r}_m)\right) = X_{-1} \cap B_{-1}(x,\mathfrak{r}_m).$$

(iv)' X_{-1} *is a* $(2r_{-1}s_{-1}+1)$*-separated* R^+_{-1}*-net in* X.

(v)' (X_{-1}, E_{-1}) *is a connected graph.*

(vi)' *We have* $d_{-1} \le d \le \lambda_{-1} d_{-1} = \Lambda_{-1} d_{-1}$.

(vii)' *We have* $\deg X_{-1} \le \Delta_{-1}$.

(viii)' *For any* $(m,x) \in \mathfrak{R}_{-1}$, *the restriction of* $\mathfrak{h}_{m,x}$ *to the subset* $X_n \cap B_{-1}(p,\mathfrak{r}_m)$ *is an* $(s_0R^+_0 + \Gamma^+_0)$*-short scale isometry with respect to* d_{-1}.

This lemma can be considered the extension to $n = -1$ of Properties (i), (ii) and (iv)–(viii) of Proposition 3.4.1. In this way, we include the case $n = 0$ in the induction step. Thus suppose that, for $n \ge 0$, we have already defined X_m, E_m, d_m and r_m for $m < n$, satisfying all the required properties. When we invoke the induction hypothesis with some item, e.g. (i), it will make reference to Lemma 3.4.2 (i)' if $n = 0$, and to Proposition 3.4.1 (i) if $n > 0$.

By (3.46), we have $\Delta_{n-1} \le \boldsymbol{\Delta}_{n-1}(\bar{\boldsymbol{r}}_{n-1})$. From this inequality, and the definitions of η_n and $\overline{\eta}_n$ in (3.12) and (3.42), we obtain, for $a \in \mathbb{N}$,

$$\eta_n(a) \ge \overline{\eta}_n(a). \tag{3.50}$$

Let $\hat{\mathfrak{c}}_n \colon X_{n-1} \to \{n, n+1, \dots\}$ be defined by

$$\begin{aligned}\hat{\mathfrak{c}}_n(x) = \min\{\, l \in \mathbb{N} \mid l \ge n,\ \exists y \in \mathfrak{X}_l \text{ so that}\\ (l,y) \in \mathfrak{R}_{n-1} \text{ and } x \in B_{-1}(y, \mathfrak{r}_l - K_{n-1}) \,\}.\end{aligned} \tag{3.51}$$

This map is well-defined because $\mathfrak{r}_l \to \infty$ as $l \to \infty$ by (3.22) and (3.24). By Proposition 3.3.4 (i), for each $x \in X_{n-1}$, there is a unique point $\hat{\mathfrak{p}}_n(x) \in \boldsymbol{\mathfrak{X}}_{\hat{\mathfrak{c}}_n(x)}$ such that $x \in B_{n-1}(\hat{\mathfrak{p}}_n(x), \mathfrak{r}_{\hat{\mathfrak{c}}_n(x)} - K_{n-1})$. This defines a map $\hat{\mathfrak{p}}_n \colon \hat{\mathfrak{c}}_n^{-1}(\{n, n+1, \dots\}) \to \boldsymbol{\mathfrak{X}}_n$.

Lemma 3.4.3. *For* $m \ge n$, *there are ordered sets* $(Y^m_n, \le^m_n)$ *such that the following properties hold:*

(a) Y^m_n *is a maximal* $2r_n$*-separated subset of* $(B_{-1}(p, \mathfrak{r}_m - K_{n-1}) \cap X_{n-1}, d_{n-1})$ *containing* p.

(b) *If $m > n$, then $Y_n^{m-1} \subset Y_n^m$, and the map $(Y_n^{m-1}, \leq_n^{m-1}) \hookrightarrow (Y_n^m, \leq_n^m)$ is order-preserving.*

(c) *For any $(l,z) \in \mathfrak{P}_{n-1}^m$, we have $\mathfrak{f}_{l,z}^m(Y_n^l) = Y_n^m \cap B_{-1}(z, \mathfrak{r}_l - K_{n-1})$, and the map*

$$\mathfrak{f}_{l,z}^m \colon \left(Y_n^l, \leq_n^l\right) \to (Y_n^m \cap B_{-1}(z, \mathfrak{r}_l - K_{n-1}), \leq_n^m)$$

is order-preserving.

(d) *For all $x, y \in Y_n^m$, we have $x <_n^m y$ if one of the following conditions holds:*

(i) $\hat{\mathfrak{c}}_n(x) < \hat{\mathfrak{c}}_n(y)$*;*

(ii) $\hat{\mathfrak{c}}_n(x) = \hat{\mathfrak{c}}_n(y)$ *and* $d_{-1}(\hat{\mathfrak{p}}_n(x), p) < d_{-1}(\hat{\mathfrak{p}}_n(y), p)$*; or*

(iii) $\hat{\mathfrak{c}}_n(x) = \hat{\mathfrak{c}}_n(y)$*,* $\hat{\mathfrak{p}}_n(x) = \hat{\mathfrak{p}}_n(y)$ *and* $d_{-1}(x, \hat{\mathfrak{p}}_n(x)) < d_{-1}(y, \hat{\mathfrak{p}}_n(x))$*.*

Proof. We proceed by induction on m. Let Y_n^n be any maximal $2r_n$-separated subset of $(B_{-1}(p, \mathfrak{r}_n - K_{n-1}) \cap X_{n-1}, d_{n-1})$ containing p. Let $\leq_n^n$ be any order relation on Y_n^n such that, if $d_{-1}(x,p) < d_{-1}(y,p)$, then $x <_n^n y$. Since $\hat{\mathfrak{c}}_n(x) = n$ and $\hat{\mathfrak{p}}_n(x) = p$ for all $x \in Y_n^n$, this relation satisfies the properties of the statement for $m = n$.

Suppose that we have defined Y_n^l and $\leq_n^l$ for $n \leq l < m$, satisfying the stated properties. Let

$$\widetilde{Y}_n^m = \bigsqcup_{(l,z) \in \overline{\mathfrak{P}}_{n-1}^m} \mathfrak{f}_{l,z}^m(Y_n^l).$$

By the induction hypothesis with (viii), for every $(l,z) \in \overline{\mathfrak{P}}_{n-1}^m$, the set $\mathfrak{h}_{l,z}(Y_n^l) = \mathfrak{f}_{l,z}^m(Y_n^l)$ is contained in X_{n-1} and is $2r_n$-separated with respect to d_{n-1}. Arguing like in the proof of Proposition 3.3.2 (i), we obtain that $\widetilde{Y}_n^m$ is a maximal $2r_n$-separated subset of

$$\bigsqcup_{(l,z) \in \overline{\mathfrak{P}}_{n-1}^m} B_{-1}(z, \mathfrak{r}_l - K_{n-1}),$$

with respect to d_{n-1}, containing p. Now, let Y_n^m be any maximal $2r_n$-separated subset of $(B_{-1}(p, \mathfrak{r}_n - K_{n-1}) \cap X_{n-1}, d_{n-1})$ containing $\widetilde{Y}_n^m$; in particular, Y_n^m satisfies (a).

Let $\widetilde{\leq}_n^m$ be any ordering of $\widetilde{Y}_n^m$ satisfying the analogues of (c), (i) and (ii) with $\widetilde{Y}_n^m$ instead of Y_n^m. Then, by the induction hypothesis with (iii) and the definition of $\widetilde{Y}_n^m$, the order $\widetilde{\leq}_n^m$ also satisfies the analogue of (iii). Let $\widehat{\leq}_n^m$ be any ordering of $\widehat{Y}_n^m := Y_n^m \setminus \widetilde{Y}_n^m$ satisfying the analogue of (iii) with $\widehat{Y}_n^m$ instead of Y_n^m. Let $\leq_n^m$ be the order relation on Y_n^m defined by $\widetilde{\leq}_n^m$ on $\widetilde{Y}_n^m$, defined by $\widehat{\leq}_n^m$ on $\widehat{Y}_n^m$, and satisfying $x \leq_n^m y$ for all $x \in \widetilde{Y}_n^m$ and $y \in Y_n^m \setminus \widetilde{Y}_n^m$. It is easy to check that $\leq_n^m$ satisfies the stated properties. □

Define $Y_n = \bigcup_{m \geq n} Y_n^m$. Like in the case of the relations $\preceq_n^m$ (Section 3.3), the order relations $\leq_n^m$ define an order relation $\leq_n$ on Y_n.

Lemma 3.4.4. *The ordered sets $(Y_n, \leq_n)$ satisfy the following properties:*

(a) *Y_n is a maximal $2r_n$-separated subset of (X_{n-1}, d_{n-1}) containing p, and therefore a $2r_n$-net in (X_{n-1}, d_{n-1}).*

(b) *For any $(l, z) \in \mathfrak{R}_{n-1}$, we have $\mathfrak{h}_{l,z}(Y_n^l) = Y_n \cap B_{-1}(z, \mathfrak{r}_l - K_{n-1})$, and the map*

$$\mathfrak{h}_{l,z} \colon (Y_n^l, \leq_n^l) \to (Y_n \cap B_{-1}(z, \mathfrak{r}_l - K_{n-1}), \leq_n)$$

is order-preserving.

(c) *For all $x, y \in Y_n$, we have $x <_n y$ if one of the following conditions holds:*

(1) $\hat{\mathfrak{c}}_n(x) < \hat{\mathfrak{c}}_n(y)$;

(2) $\hat{\mathfrak{c}}_n(x) = \hat{\mathfrak{c}}_n(y)$ *and* $d_{-1}(\hat{\mathfrak{p}}_n(x), p) < d_{-1}(\hat{\mathfrak{p}}_n(y), p)$; *or*

(3) $\hat{\mathfrak{c}}_n(x) = \hat{\mathfrak{c}}_n(y)$, $\hat{\mathfrak{p}}_n(x) = \hat{\mathfrak{p}}_n(y)$ *and* $d_{-1}(x, \hat{\mathfrak{p}}_n(x)) < d_{-1}(y, \hat{\mathfrak{p}}_n(x))$.

(d) *$(Y_n, \leq_n)$ is well-ordered.*

Proof. Properties (a)–(c) follow from Lemma 3.4.3 (a)–(c) and the definition of $(Y_n, \leq_n)$. So let us prove (d). By (1), it is enough to prove that, for each $m \geq n$, the ordered subset $(Y_n \cap \hat{\mathfrak{c}}_n^{-1}(m), \leq_n)$ is well-ordered. By (2), the subsets $\{\, y \in Y_n \cap \hat{\mathfrak{c}}_n^{-1}(m) \mid d_{-1}(\hat{\mathfrak{p}}(y), p) \leq l \,\}$, with $l \in \mathbb{N}$, form an increasing sequence of finite initial segments[2] of $(Y_n \cap \hat{\mathfrak{c}}_n^{-1}(m), \leq_n)$ covering $Y_n \cap \hat{\mathfrak{c}}_n^{-1}(m)$. Since

$$\begin{aligned}\{\, y \in Y_n \cap \hat{\mathfrak{c}}_n^{-1}(m) \mid d_{-1}(\hat{\mathfrak{p}}(y), p) \leq l \,\} &\subset \bigcup_{y \in Y_n,\ d_{-1}(y,p) \leq l} B_{-1}(y, \mathfrak{r}_m - K_{n-1}) \\ &\subset B_{-1}(p, l + \mathfrak{r}_m - K_{n-1}),\end{aligned}$$

all sets $\{\, y \in Y_n \cap \hat{\mathfrak{c}}_n^{-1}(m) \mid d_{-1}(\hat{\mathfrak{p}}(y), p) \leq l \,\}$ are finite, and therefore well-ordered with $\leq_n$. Then it easily follows that $Y_n \cap \hat{\mathfrak{c}}_n^{-1}(m)$ is well-ordered, completing the proof of (d). □

Remark 3.4.3. Note that $\{n\} \times \mathfrak{X}_n \subset \mathfrak{R}_{n-1}$ by definition. By Lemma 3.4.4 (a),(b), for any $x \in \mathfrak{X}_n$, we have $x = \mathfrak{h}_{n,x}(p) \subset Y_n$, yielding $\mathfrak{X}_n \subset Y_n$.

Remark 3.4.4. For any $x \in B_{-1}(p, \mathfrak{r}_m - K_{n-1})$, we have $\hat{\mathfrak{c}}_n(x) = n$ and $\hat{\mathfrak{p}}_n(x) = p$ by definition. So, by (2), $B_{-1}(p, \mathfrak{r}_m - K_{n-1})$ is an initial segment of Y_n. Therefore p is the least element of Y_n by (3).

Let now

$$\begin{aligned} Y_n^- &= \{\, y \in Y_n \mid \eta_n(|B_{n-1}(y, r_n^+)|) < (6 + |B_{n-1}(y, r_n^+ s_n)|)^2 \,\}, \\ Y_n^+ &= \{\, y \in Y_n \mid \eta_n(|B_{n-1}(y, r_n^+)|) \geq (6 + |B_{n-1}(y, r_n^+ s_n)|)^2 \,\}. \end{aligned}$$

[2]For an ordered set $(A, \leq)$, an *initial segment* of $(A, \leq)$ is a subset that is of the form $\{\, x \in A \mid x \leq a \,\}$ for some $a \in A$.

Lemma 3.4.5. *If $y \in B_{-1}(p, \mathfrak{r}_l - K_{n-1} - \Lambda_{n-1} r_n s_n^2)$, then $B_{n-1}(y, r_n^+ s_n) \subset B_{-1}(p, \mathfrak{r}_l - K_{n-1})$.*

Proof. By the induction hypothesis with Proposition 3.4.1 (vi), we have

$$\begin{aligned} d_{-1}(x,p) &\leq d_{-1}(x,y) + d_{-1}(y,p) \leq \Lambda_{n-1} d_{n-1}(x,y) + d_{n-1}(y,p) \\ &\leq \Lambda_{n-1} r_n s_n^2 + \mathfrak{r}_l - K_{n-1} - \Lambda_{n-1} r_n s_n^2 = \mathfrak{r}_l - K_{n-1}. \end{aligned}$$ □

Lemma 3.4.6. *For any $(l,z) \in \mathfrak{R}_{n-1}$ and $y \in Y_n \cap B_{-1}(p, \mathfrak{r}_l - K_{n-1} - \Lambda_{n-1} r_n s_n^2)$, we have that $y \in Y_n^{\pm}$ if and only if $\mathfrak{h}_{l,z}(y) \in Y_n^{\pm}$.*

Proof. By Lemma 3.4.5, we have $B_{n-1}(y, r_n s_n^2) \subset B_{-1}(p, \mathfrak{r}_l - K_{n-1}) \subset \operatorname{dom}(\mathfrak{h}_{l,z})$. Since $\mathfrak{h}_{l,z}$ is a $s_n R_n^+$-short scale isometry over $(B_{-1}(p, \mathfrak{r}_l - K_{n-1}), d_{n-1})$, we get $|B_{n-1}(y, r_n s_n^i)| = |B_{n-1}(\mathfrak{h}_{l,z}(y), r_n s_n^i)|$ for $i = 1, 2$. □

Using that $(Y_n, \leq_n)$ is a well-ordered set (Lemma 3.4.4 (d)), let $X_n^+ \subset Y_n^+$ be the subset inductively defined as follows:

- If y_0 is the least element of $(Y_n^+, \leq_n)$, then $y_0 \in X_n^+$.
- For all $y \in Y_n^+$ such that $y >_n y_0$, we have $y \in X_n^+$ if and only if, for any $y' \in X_n^+$ with $y' <_n y$, we have $d_{n-1}(y, y') > 2r_n s_n$.

Remark 3.4.5. Observe that X_n^+ is a $(2r_n s_n + 1)$-separated $2r_n s_n$-net in (Y_n^+, d_{n-1}).

Remark 3.4.6. Note that Lemma 3.4.4 (b) yields $Y_n^l = Y_n \cap B_{-1}(p, \mathfrak{r}_l - K_{n-1})$ because $\mathfrak{h}_{l,p} = \mathrm{id}$ by Proposition 3.3.4 (vi).

Lemma 3.4.7. *For all $z \in \mathfrak{X}_n$ and $y \in Y_n \cap B_{-1}(p, \mathfrak{r}_n - K_{n-1} - \Lambda_{n-1} r_n s_n^2)$, we have $y \in X_n^+$ if and only if $\mathfrak{h}_{n,z}(y) \in X_n^+$.*

Proof. By Lemma 3.4.6, it is enough to prove the statement for points $y \in Y_n^+$. We proceed by induction on the elements of $Y_n^+ \cap B_{-1}(p, \mathfrak{r}_n - K_{n-1} - \Lambda_{n-1} r_n s_n^2)$ using $\leq_n$. Let y_1 be the least element of $Y_n^+ \cap B_{-1}(p, \mathfrak{r}_n - K_{n-1} - \Lambda_{n-1} r_n s_n^2)$. We first prove that $y_1, \mathfrak{h}_{n,z}(y_1) \in X_n^+$, establishing the desired property for y_1.

By absurdity, suppose that $y_1 \notin X_n^+$. This means that $y_1 >_n y_0$ and there is some $u \in X_n^+$ such that $u <_n y_1$ and $d_{n-1}(y_1, u) \leq 2r_n s_n$. Since $s_n > 2$ by (3.3) and (3.11), it follows from Lemma 3.4.5 that $u \in B_{-1}(p, \mathfrak{r}_n - K_{n-1})$. Then $\hat{\mathfrak{c}}_n(y_1) = \hat{\mathfrak{c}}_n(u) = n$ and $\hat{\mathfrak{p}}_n(y_1) = \hat{\mathfrak{p}}_n(u) = p$. Lemma 3.4.4 (3) and the assumption that $u <_n y_1$ yield $d_{-1}(p, u) \leq d_{-1}(p, y_1)$. So, in fact, $u \in B_{-1}(p, \mathfrak{r}_n - K_{n-1} - \Lambda_{n-1} r_n s_n^2)$, contradicting the hypothesis that y_1 is the least element of $B_{-1}(p, \mathfrak{r}_n - K_{n-1} - \Lambda_{n-1} r_n s_n^2)$. This shows that $y_1 \in X_n^+$.

By Lemma 3.4.4 (b) and Remark 3.4.6, the map $\mathfrak{h}_{n,z}$ preserves $\leq_n$ over $B_{-1}(p, \mathfrak{r}_n - K_{n-1})$. So, using the same argument, we get $\mathfrak{h}_{n,z}(y_1) \in X_n^+$.

Now, given $y \in Y_n^+ \cap B_{-1}(p, \mathfrak{r}_n - K_{n-1} - \Lambda_{n-1} r_n s_n^2)$ so that $y_1 <_n y$, suppose that the result is true for all $y' \in Y_n^+ \cap B_{-1}(p, \mathfrak{r}_n - K_{n-1} - \Lambda_{n-1} r_n s_n^2)$ with $y' <_n y$. By definition, we have $y \notin X_n^+$ if and only if there is some $u \in X_n^+$ such that $u <_n y$ and $d_{n-1}(u,p) \leq 2 r_n s_n$. Using the same argument as before, we obtain that, necessarily, $u \in B_{-1}(p, \mathfrak{r}_n - K_{n-1} - \Lambda_{n-1} r_n s_n^2)$. By the induction hypothesis, we have $\mathfrak{h}_{n,z}(u) \in X_n^+$. Then $y \notin X_n^+$ if and only if there is some $u \in B_{-1}(\mathfrak{r}_n - K_{n-1})$ with $\mathfrak{h}_{n,z}(u) \in X_n^+$ and $d_{n-1}(\mathfrak{h}_{n,z}(u), \mathfrak{h}_{n,z}(y)) \leq 2 r_n s_n$. But, by the induction hypothesis with (viii), we have $d_{n-1}(\mathfrak{h}_{n,z}(u), \mathfrak{h}_{n,z}(y)) = d_{n-1}(u,y) \leq 2 r_n s_n$. So $y \in X_n^+$ if and only if $\mathfrak{h}_{n,z}(y) \in X_n^+$, as desired. □

Proposition 3.4.8. *For all $(l,z) \in \mathfrak{R}_{n-1}$ and $y \in Y_n \cap B_{-1}(p, \mathfrak{r}_l - K_{n-1} - \Lambda_{n-1} r_n s_n^2)$, we have $y \in X_n^+$ if and only if $\mathfrak{h}_{l,z}(y) \in X_n^+$.*

Proof. We proceed by induction on $l \geq n$. The case $l = n$ is precisely the statement of Lemma 3.4.7. Therefore take any $l > n$ and suppose that the result is true for $n \leq l' < l$.

By Lemma 3.4.6, it is enough to prove the statement for points $y \in Y_n^+$. We proceed by induction on the elements of $Y_n^+ \cap B_{-1}(p, \mathfrak{r}_l - K_{n-1} - \Lambda_{n-1} r_n s_n^2)$ using $\leq_n$. Let y_1 be the least element of $Y_n^+ \cap B_{-1}(p, \mathfrak{r}_l - K_{n-1} - \Lambda_{n-1} r_n s_n^2)$. We will prove that $y_1 \notin X_n^+$ if and only if $\mathfrak{h}_{l,z}(y_1) \notin X_n^+$, establishing the desired property for y_1.

The condition $y_1 \notin X_n^+$ means that $y_1 >_n y_0$ and there is some $u \in X_n^+$ such that $u <_n y_1$ and $d_{n-1}(y_1, u) \leq 2 r_n s_n$. Since $s_n > 2$ by (3.3) and (3.11), it follows from Lemma 3.4.5 that $u \in B_{-1}(p, \mathfrak{r}_l - K_{n-1})$, and therefore $\hat{\mathfrak{c}}(y_1), \hat{\mathfrak{c}}(u) \leq l$. We will consider several cases about u.

Suppose that $\hat{\mathfrak{c}}_n(u) > \hat{\mathfrak{c}}_n(y_1)$. Then $y_1 <_n u$ by Lemma 3.4.4 (1), contradicting the assumption that $u <_n y_1$.

Suppose then that $\hat{\mathfrak{c}}(y_1) = \hat{\mathfrak{c}}(u) = l$. Thus $\hat{\mathfrak{p}}(y_1) = \hat{\mathfrak{p}}(u) = p$. Lemma 3.4.4 (3) and the assumption that $u <_n y_1$ yield $d_{-1}(p,u) \leq d_{-1}(p, y_1)$. Therefore $u \in Y_n^+ \cap B_{-1}(p, \mathfrak{r}_l - K_{n-1} - \Lambda_{n-1} r_n s_n^2)$, contradicting the hypothesis that y_1 is the least element in $Y_n^+ \cap B_{-1}(p, \mathfrak{r}_l - K_{n-1} - \Lambda_{n-1} r_n s_n^2)$.

Suppose finally that $\hat{\mathfrak{c}}(u) < l$. Then $\mathfrak{h}_{\hat{\mathfrak{c}}(u), \hat{\mathfrak{p}}(u)}(u) \in X_n^+$ by the induction hypothesis with l. But, by the induction hypothesis with (viii), we have $d_{n-1}(\mathfrak{h}_{l,z}(u), \mathfrak{h}_{l,z}(y_1)) = d_{n-1}(u, y_1) \leq 2 r_n s_n$. So $\mathfrak{h}_{l,z}(y_1) \notin X_n^+$.

Thus far, we have proved that $y_1 \notin X_n^+$ implies $\mathfrak{h}_{l,z}(y_1) \notin X_n^+$. The proof of the converse implication is similar

Now, given $y \in Y_n^+ \cap B_{-1}(p, \mathfrak{r}_l - K_{n-1} - \Lambda_{n-1} r_n s_n^2)$ so that $y_1 <_n y$, suppose that the result is true for all $y' \in Y_n^+ \cap B_{-1}(p, \mathfrak{r}_l - K_{n-1} - \Lambda_{n-1} r_n s_n^2)$ with $y' <_n y$. By definition, we have that $y \notin X_n^+$ if and only if there is some $u \in X_n^+$ such that $u <_n y$ and $d_{n-1}(u,p) \leq 2r_n s_n$. Using the same argument as before, we obtain that, either $\hat{\mathfrak{c}}_n(u) < l$, or $u \in B_{-1}(p, \mathfrak{r}_l - K_{n-1} - \Lambda_{n-1} r_n s_n^2)$. If $\hat{\mathfrak{c}}_n(u) < l$, we get $\mathfrak{h}_{l,z}(y) \notin X_n^+$ arguing as before. If $u \in B_{-1}(p, \mathfrak{r}_l - K_{n-1} - \Lambda_{n-1} r_n s_n^2)$, then $\mathfrak{h}_{l,z}(u) \in X_n^+$ by the induction hypothesis in $Y_n^+ \cap B_{-1}(p, \mathfrak{r}_l - K_{n-1} - \Lambda_{n-1} r_n s_n^2)$. Thus $y \notin X_n^+$ if and only if there is some $u \in B_{-1}(\mathfrak{r}_l - K_{n-1})$ with $\mathfrak{h}_{l,z}(u) \in X_n^+$ and $d_{n-1}(\mathfrak{h}_{l,z}(u), \mathfrak{h}_{l,z}(y)) \leq 2r_n s_n$. But $d_{n-1}(\mathfrak{h}_{l,z}(u), \mathfrak{h}_{l,z}(y)) = d_{n-1}(u,y) \leq 2r_n s_n$ by the induction hypothesis with (viii). So $y \in X_n^+$ if and only if $\mathfrak{h}_{l,z}(y) \in X_n^+$, as desired. □

Let

$$X_n^- = \{\, y \in Y_n^- \mid d_{n-1}(y, X_n^+) > r_n(2s_n + 1) \,\}. \tag{3.52}$$

Lemma 3.4.9. *We have $p \in X_n$.*

Proof. Suppose first that condition (A) is satisfied in the definition of r_n, and consequently $r_n = \bar{r}_n$. Then there is some point $x \in B_{n-1}(p, \hat{r}_n(2s_n + 1))$ such that

$$(|B_{n-1}(x, \hat{r}_n s_n)| + 6)^2 \geq \eta_n(|B_{n-1}(x, \hat{r}_n)|). \tag{3.53}$$

So $B_{n-1}(x, \hat{r}_n s_n) \subset B_{n-1}(p, \hat{r}_n(3s_n + 1))$, and therefore

$$|B_{n-1}(p, r_n)| = |B_{n-1}(p, \hat{r}_n(3s_n + 1))| \geq |B_{n-1}(x, \hat{r}_n s_n)|. \tag{3.54}$$

Using (3.14), (3.15), (3.50), (3.53) and (3.54), we get

$$\begin{aligned}
\eta_n(|B_{n-1}(p, r_n s_n)|) &\geq \eta_n(|B_{n-1}(x, \hat{r}_n s_n)|) \geq \eta_n\big(\sqrt{\eta_n(|B_{n-1}(x, \hat{r}_n)|)} - 6\big) \\
&\geq \overline{\eta}_n\big(\sqrt{\overline{\eta}_n(|B_{n-1}(x, \hat{r}_n)|)} - 6\big) > \overline{\eta}_n\big(\sqrt{\overline{\eta}_n(\hat{r}_n)} - 6\big) \\
&\geq \Big(4(\boldsymbol{\Delta}_{n-1}(\bar{\boldsymbol{r}}_{n-1}) - 1)^{\bar{r}_n s_n^2} + 6\Big)^2 .
\end{aligned}$$

The assumption $r_n = \bar{r}_n$ implies $\bar{\boldsymbol{r}}_{n-1} = (r_0, \ldots, r_{n-1})$ and $\boldsymbol{\Delta}_{n-1}(\bar{\boldsymbol{r}}_{n-1}) = \Delta_{n-1}$ according to (3.44). Hence, by (2.5),

$$\begin{aligned}
\eta_n(|B_{n-1}(p, r_n s_n)|) &\geq \Big(4(\boldsymbol{\Delta}_{n-1}(\bar{\boldsymbol{r}}_{n-1}) - 1)^{\bar{r}_n s_n^2} + 6\Big)^2 \\
&= \Big(4(\Delta_{n-1} - 1)^{r_n s_n^2} + 6\Big)^2 \geq (|B_{n-1}(p, r_n s_n^2)| + 6)^2,
\end{aligned}$$

and therefore $p \in Y_n^+$. Then the lemma follows in this case from Remark 3.4.4 and the definition of X_n^+.

Suppose now that condition (B) holds. Then $p \in Y_n^-$ and $Y_n^+ \cap B_{n-1}(p, r_n(2s_n + 1)) = \emptyset$, and the lemma also follows in this second case. □

By (3.20), (3.21), (3.44) and (3.45), we have

$$\overline{K}_n = K_{n-1} + \Lambda_n(r_n s_n^2 + r_n(2s_n + 1)), \tag{3.55}$$
$$K_n = \overline{K}_n + \Lambda_n(s_{n+1}R_{n+1}^+ + \Gamma_n^+ + 2R_n^+). \tag{3.56}$$

Lemma 3.4.10. *For all* $(l, z) \in \mathfrak{R}_{n-1}$ *and* $y \in Y_n \cap B_{-1}(p, \mathfrak{r}_l - \overline{K}_n)$*, we have* $y \in X_n^-$ *if and only if* $\mathfrak{h}_{l,z}(y) \in X_n^-$*.*

Proof. Let $y \in Y_n \cap B_{-1}(p, \mathfrak{r}_l - \overline{K}_n)$. Then, by (3.55),

$$y \in Y_n \cap B_{-1}(p, \mathfrak{r}_l - K_{n-1} - \Lambda_{n-1}(r_n s_n^2 + r_n(2s_n + 1))).$$

By Lemma 3.4.6, we can assume $y, \mathfrak{h}_{l,z}(y) \in Y_n^-$. Hence, by definition, $y \notin X_n^-$ if and only if there is some $x \in X_n^+$ with $d_{n-1}(y, x) \le r_n(2s_n + 1)$. In this case, by the induction hypothesis with (vi), we have $d_{-1}(x, y) \le \Lambda_{n-1} r_n(2s_n + 1)$. Therefore, by the triangle inequality, $x \in B_{-1}(p, \mathfrak{r}_l - K_{n-1} - \Lambda_{n-1} r_n s_n^2) \subset B_{-1}(p, \mathfrak{r}_m - K_n)$. Applying now Proposition 3.4.8, we get $\mathfrak{h}_{l,z}(x) \in X_n^+$. Also, by the induction hypothesis with (viii), $\mathfrak{h}_{l,z}$ is a $s_n R_n^+$-short scale isometry on $(X_{n-1} \cap B_{-1}(p, \mathfrak{r}_m - K_n), d_{n-1})$. Therefore $\mathfrak{h}_{l,z}(x) \in X_n^+$ and $d_{n-1}(\mathfrak{h}_{l,z}(x), \mathfrak{h}_{l,z}(y)) \le r_n(2s_n + 1)$, obtaining $\mathfrak{h}_{l,z}(y) \notin X_n^-$.

The proof of the converse implication is similar. □

Let us prove (i). By Lemma 3.4.9, we have $p \in X_n$ and $(n, x) \in \mathfrak{R}_{n-1}$ for each $x \in \mathfrak{X}_n$,. Proposition 3.4.8 and Lemma 3.4.10 then imply $x = \mathfrak{h}_{n,x}(p) \in X_n$ for all $x \in \mathfrak{X}_n$, obtaining $\mathfrak{X}_n \subset X_n$. The inclusion $X_n \subset X_{n-1}$ follows from Lemma 3.4.4 (a) and the fact that $X_n \subset Y_n$. This completes the proof of (i).

For all $(m, x) \in \mathfrak{R}_{n-1}$, the map $\mathfrak{h}_{m,x} \colon (B_{-1}(p, \mathfrak{r}_m), p) \to (B_{-1}(x, \mathfrak{r}_m), x)$ is a pointed isometry by definition. Therefore $\mathfrak{h}_{m,x}(B_{-1}(p, \mathfrak{r}_m - \overline{K}_n)) = B_{-1}(x, \mathfrak{r}_m - \overline{K}_n)$. Then property (ii) follows from Proposition 3.4.8 and Lemma 3.4.10.

Let us prove (iii). For $x \in X_n^+$, the result is an immediate consequence of the definition of Y_n^+ and the fact that $X_n^+ \subset Y_n^+$. So assume $x \in X_n^-$. By absurdity, suppose that $(|B_{n-1}(x, r_n s_n)| + 6)^2 > \eta_n(|B_{n-1}(x, r_n)|)$. Since η_n is an increasing function, and using (3.50), (3.15), (3.44) and (2.5), we get

$$\begin{aligned}
\eta_n(|B_{n-1}(x, r_n s_n)|) &\ge \eta_n\left(\sqrt{\eta_n(|B_{n-1}(x, r_n)|)} - 6\right) \ge \overline{\eta}_n\left(\sqrt{\overline{\eta}_n(|B_{n-1}(x, r_n)|)} - 6\right) \\
&> \overline{\eta}_n\left(\sqrt{\overline{\eta}_n(r_n)} - 6\right) > \left(4(\boldsymbol{\Delta}_{n-1}(\bar{\boldsymbol{r}}_{n-1}) - 1)^{\bar{r}_n s_n^2} + 6\right)^2 \\
&= \left(4(\Delta_{n-1} - 1)^{\bar{r}_n s_n^2} + 6\right)^2 \ge (|B_{n-1}(x, r_n s_n^2)| + 6)^2.
\end{aligned}$$

Therefore $x \notin Y_n^-$ by definition, contradicting the assumption that $x \in X_n^-$, which completes the proof of (iii).

Let us prove (iv). First, define

$$\mathcal{Z}_{n-1}^- = \{\, z \in X_{n-1} \mid d_{n-1}(z, X_n^+) - 2r_n s_n > d_{n-1}(z, X_n^-) - r_n \,\}, \tag{3.57}$$
$$\mathcal{Z}_{n-1}^+ = \{\, z \in X_{n-1} \mid d_{n-1}(z, X_n^+) - 2r_n s_n \leq d_{n-1}(z, X_n^-) - r_n \,\}. \tag{3.58}$$

Thus $X_{n-1} = \mathcal{Z}_{n-1}^- \sqcup \mathcal{Z}_{n-1}^+$. On the other hand, using (3.9), (3.16), (3.41) and (3.45), we get

$$R_n^- = 4r_n - 1 \ , \quad R_n^+ = r_n(2s_n + 3).$$

Lemma 3.4.11. *X_n^+ is a $(2r_n s_n + 1)$-separated R_n^+-net in $(\mathcal{Z}_{n-1}^+, d_{n-1})$.*

Proof. By Remark 3.4.5, we only need to show that X_n^+ is an R_n^+-net in $(\mathcal{Z}_{n-1}^+, d_{n-1})$. Take an arbitrary point $z \in \mathcal{Z}_{n-1}^+$. Since Y_n is a $2r_n$-net in (X_{n-1}, d_{n-1}) by Lemma 3.4.4 (a), there is some $y \in Y_n$ with $d_{n-1}(x, z) \leq 2r_n$.

If $y \in Y_n^+$, then, by Remark 3.4.5, there is some $x \in X_n^+$ with $d_{n-1}(y, x) \leq 2r_n s_n$. Using the triangle inequality, we get

$$d(z, x) \leq d(z, y) + d(y, x) \leq 2r_n + 2r_n s_n < r_n(2s_n + 3) = R_n^+.$$

If $y \in X_n^-$, we have $d_{n-1}(z, X_n^-) \leq 2r_n$. Then (3.58) implies $d_{n-1}(z, X_n^+) - 2r_n s_n \leq r_n$, obtaining $d_{n-1}(z, X_n^+) \leq r_n(2s_n + 1) < R_n^+$.

Finally, suppose that $y \in Y_n^- \setminus X_n^-$. By (3.52), there is some point $x \in X_n^+$ with $d_{n-1}(x, y) \leq r_n(2s_n + 1)$, and the lemma follows applying the triangle inequality:

$$d(z, x) \leq d(z, y) + d(y, x) \leq 2r_n + r_n(2s_n + 1) = r_n(2s_n + 3) = R_n^+. \qquad \square$$

Lemma 3.4.12. *X_n^- is a $(2r_n s_n + 1)$-separated R_n^--net in $(\mathcal{Z}_{n-1}^-, d_{n-1})$.*

Proof. Let $z \in \mathcal{Z}_{n-1}^-$. Like in Lemma 3.4.11, there is some $y \in Y_n$ with $d_{n-1}(z, y) \leq 2r_n$.

In the case where $y \in X_n^-$, the lemma is trivial.

If $y \in X_n^+$, then $d_{n-1}(z, X_n^+) \leq 2r_n$, yielding $d_{n-1}(z, X_n^+) - 2r_n s_n \leq 2r_n(1 - s_n)$. Using (3.57), we get $d_{n-1}(y, X_n^-) - r_n < 2r_n(1 - s_n)$, and therefore $d_{n-1}(y, X_n^-) < 2r_n(2 - s_n)$. However, by (3.3) and (3.11), we have $s_n > 2$, reaching a contradiction. Therefore $y \notin X_n^+$.

Now, suppose $y \in Y_n^+ \setminus X_n^+$. By Remark 3.4.5, there is some $x \in X_n^+$ with $d_{n-1}(x, y) \leq 2r_n s_n$, and we get $d_{n-1}(z, x) \leq 2r_n(s_n + 1)$ using the triangle inequality. Then (3.57) yields

$$\begin{aligned} d_{n-1}(z, X_n^-) &< d_{n-1}(z, X_n^+) - 2r_n s_n + r_n \leq d_{n-1}(z, x) - 2r_n s_n + r_n \\ &\leq 2r_n(s_n + 1) - 2r_n s_n + r_n = 3r_n \leq R_n^-. \end{aligned}$$

Finally, suppose $y \in Y_n^- \setminus X_n^-$. By (3.52), there is some point $x \in X_n^+$ with $d_{n-1}(x, y) \leq r_n(2s_n + 1)$, obtaining $d_{n-1}(z, X_n^+) \leq r_n(2s_n + 3)$ by the triangle inequality. Therefore $d_{n-1}(z, X_n^+) - 2r_n s_n \leq 3r_n$, obtaining $d_{n-1}(z, X_n^-) < 4r_n$ by (3.57); i.e., $d_{n-1}(z, X_n^-) \leq 4r_n - 1 = R_n^-$. $\square$

To complete the proof of Proposition 3.4.1 (iv), it only remains to prove that

$$d_{n-1}(X_n^-, X_n^+) \geq 2r_n s_n + 1,$$

which follows from (3.52).

To prove the next items of Proposition 3.4.1, we need some more preliminary results.

Lemma 3.4.13. *For all $z \in X_{n-1}$, we have $z \in \mathcal{Z}_{n-1}^+$ if and only if*

$$d_{n-1}(z, X_n^+ \cap B_{n-1}(z, R_n^+)) - 2r_n s_n \leq d_{n-1}(z, X_n^- \cap B_{n-1}(z, R_n^+)) - r_n. \tag{3.59}$$

Proof. Suppose first that $z \in \mathcal{Z}_{n-1}^+$. Lemma 3.4.11 implies $X_n^+ \cap B_{n-1}(z, R_n^+) \neq \emptyset$, and therefore

$$d_{n-1}(z, X_n^+ \cap B_{n-1}(z, R_n^+)) = d_{n-1}(z, X_n^+).$$

Then (3.57) implies (3.59).

Suppose now that (3.59) holds for some $z \in X_{n-1}$. Property (iv) implies that at least one of the inequalities $d_{n-1}(z, X_n^-) \leq R_n^+$ or $d_{n-1}(z, X_n^+) \leq R_n^+$ is satisfied. So at least the left-hand side of (3.59) is finite. Therefore (3.59) yields (3.57). □

Corollary 3.4.14. *For all $u \in X_{n-1} \cap B_{-1}(p, \mathfrak{r}_l - \overline{K}_n - \Lambda_{n-1} R_n^+)$ and $(l, z) \in \mathfrak{R}_{n-1}$, we have $u \in \mathcal{Z}_{n-1}^\pm$ if and only if $\mathfrak{h}_{l,z}(u) \in \mathcal{Z}_{n-1}^\pm$.*

Proof. Let $u \in X_{n-1} \cap B_{-1}(p, \mathfrak{r}_l - \overline{K}_n - \Lambda_{n-1} R_n^+)$ and $(l, z) \in \mathfrak{R}_{n-1}$. Since $X_{n-1} = \mathcal{Z}_{n-1}^- \sqcup \mathcal{Z}_{n-1}^+$, it is enough to prove that $u \in \mathcal{Z}_{n-1}^+$ if and only if $\mathfrak{h}_{l,z}(u) \in \mathcal{Z}_{n-1}^+$.

We have $B_{n-1}(u, R_n^+) \subset B_{-1}(p, \mathfrak{r}_l - \overline{K}_n) \subset \operatorname{dom}(\mathfrak{h}_{l,z})$ by the induction hypothesis with (vi) and the triangle inequality, . Proposition 3.4.8 and Lemma 3.4.10, and the induction hypothesis with (viii) imply that the restriction of $\mathfrak{h}_{l,z}$ to $B_{-1}(p, \mathfrak{r}_l - \overline{K}_n)$ preserves $X_n^\pm$ and is an R_n^+-partial isometry with respect to d_{n-1}. Then the result follows from Lemma 3.4.13. □

Remark 3.4.7. Note that (3.55) yields $K_n \geq \overline{K}_n + \Lambda_{n-1} R_n^+$. Then $\mathfrak{r}_l - \overline{K}_n - \Lambda_{n-1} R_n^+ > 0$ in Corollary 3.4.14 by (3.22).

Recall the definition of $r_n^\pm$ given in (3.48).

Lemma 3.4.15. *If $x \in X_n^\pm$, then $B_{n-1}(x, r_n^\pm) \subset \mathcal{Z}_{n-1}^\pm$.*

Proof. For $x \in X_n^-$, suppose on the contrary that there is some $z \in B_{n-1}(x, r_n)$ such that

$$d_{n-1}(z, X_n^+) - 2r_n s_n \leq d_{n-1}(z, X_n^-) - r_n.$$

In particular, $d_{n-1}(z, X_n^+) \leq 2r_n s_n$ because $d_{n-1}(z, X_n^-) \leq d_{n-1}(z, x) \leq r_n$. By the triangle inequality, it follows that

$$d_{n-1}(x, X_n^+) \leq d_{n-1}(x, z) + d_{n-1}(z, X_n^+) \leq r_n + 2r_n s_n = r_n(2s_n + 1),$$

contradicting the definition of X_n^- in (3.52).

The proof when $x \in X_n^+$ is similar. □

For every $x \in X_n$, let

$$\overline{C}_{n,n-1}(x) = \{\, z \in \mathcal{Z}_{n-1}^{\pm} \mid d_{n-1}(z,x) = d_{n-1}(z, X_n^{\pm}) \,\} \quad \text{if} \quad x \in X_n^{\pm}. \tag{3.60}$$

Remark 3.4.8. Observe that the sets $\overline{C}_{n,n-1}(x)$, for $x \in X_n$, cover X_{n-1}.

Lemma 3.4.16. *For $x \in X_n^{\pm}$, we have $\overline{C}_{n,n-1}(x) \subset B_{n-1}(x, R_n^{\pm})$.*

Proof. This is a direct consequence of Lemmas 3.4.11 and 3.4.12. □

Define a graph structure E_n on X_n by declaring that there is an edge between $x, y \in X_n$ if

$$d_{n-1}(\overline{C}_{n,n-1}(x), \overline{C}_{n,n-1}(y)) \le 1. \tag{3.61}$$

To prove (v), consider two points $x, y \in X_n$. By the induction hypothesis with (v), X_{n-1} is connected, and, by construction, $X_n \subset X_{n-1}$. So there must be some path in (X_{n-1}, E_{n-1}) of the form $(u_0 = x, u_1, \dots, u_a = y)$. By Remark 3.4.8, for each $i = 0, \dots, a$, there is some $z_i \in X_n$ such that $u_i \in \overline{C}_{n,n-1}(z_i)$, $z_0 = x$ and $z_a = y$. Clearly,

$$d_{n-1}(\overline{C}_{n,n-1}(z_{i-1}), \overline{C}_{n,n-1}(z_i)) \le 1$$

for $i = 1, \dots, a$. Therefore $(z_0, \dots, z_a)$ is a path in X_n connecting x to y.

Let us prove (vi). For any $x, y \in X_n$ with $d_n(x,y) = a$, there is a sequence $(x_0 = x, x_1, \dots, x_a = y)$ in X_n such that $d_n(\overline{C}_{n,n-1}(x_{i-1}), \overline{C}_{n,n-1}(x_i)) \le 1$ for each $i = 1, \dots, a$. By Lemma 3.4.16, (3.9) and (3.45), we have $d_{n-1}(x_{i-1}, x_i) \le 2R_n^+ + 1 = \lambda_n$. Then (vi) follows from the triangle inequality, using (3.18), (3.41) and (3.44).

Let us prove (vii). For $x, y \in X_n$, if $x E_n y$, then $d_{n-1}(x,y) \le 2R_n^+ + 1$ by (3.61) and Lemma 3.4.16. So

$$|S_n(x,1)| \le |B_{n-1}(x, 2R_n^+ + 1)| \le 4(\deg X_{n-1} - 1)^{2R_n^+}$$

by (2.5). Then the bound $\deg X_n \le \Delta_n$ follows by induction with (vii), using (3.10), (3.17) and (3.44).

Let us prove (viii). Let $(m,z) \in \mathfrak{R}_{n-1}$ and $x \in X_n \cap B_{-1}(p, \mathfrak{r}_m - \overline{K}_n - 2\Lambda_n R_n^+)$. Then

$$\overline{C}_{n,n-1}(x) \subset B_{-1}(p, \mathfrak{r}_m - \overline{K}_n - \Lambda_n R_n^+) \subset \operatorname{dom}(\mathfrak{h}_{m,z}) \tag{3.62}$$

by Lemma 3.4.16, Proposition 3.3.4 (v), and the induction hypothesis with (vi) and (viii). Recall that $\mathfrak{R}_{n-1} \subset \mathfrak{R}_{n-2}$ by (3.36) and (3.40). Furthermore, from the induction hypothesis with (viii), Proposition 3.3.4 (v), Corollary 3.4.14, (3.60) and (3.62), it follows that

$$\mathfrak{h}_{m,z}\left(\overline{C}_{n,n-1}(x)\right) = \overline{C}_{n,n-1}\left(\mathfrak{h}_{m,z}(x)\right). \tag{3.63}$$

So, for $x, y \in X_n \cap B_{-1}(p, \mathfrak{r}_m - \overline{K}_n - 2\Lambda_n R_n^+)$, (3.61) holds if and only if

$$d_{n-1}(\overline{C}_{n,n-1}(\mathfrak{h}_{m,z}(x)), \overline{C}_{n,n-1}(\mathfrak{h}_{m,z}(y))) \leq 1.$$

Therefore xE_ny if and only if $\mathfrak{h}_{m,z}(x)E_n\mathfrak{h}_{m,z}(y)$. Then property (viii) follows from the induction hypothesis with (vi), Lemma 2.2.15 and (3.56).

3.5 Clusters

In order to define the colorings satisfying the conditions of Theorems (3.1.1) and (3.1.2), we will divide the sets X_{n-1} into "clusters", denoted by $C_{n,n-1}(x)$ and indexed by $x \in X_n$. These will be used in Section 3.6 to construct the suitable colorings locally on this family of sets.

In Section 3.4, we have defined well-ordered sets $(Y_n, \leq_n)$ for $n \in \mathbb{N}$, whose restrictions to the subset X_n determine a family of well-orders $\leq_n$. For $n \in \mathbb{N}$, let $\pi_n^\pm \colon \mathcal{Z}_{n-1}^\pm \to X_n^\pm$ be defined by

$$\pi_{n-1}^\pm(u) = \inf\{\, x \in X_n^\pm \mid d_{n-1}(u, x) = d_{n-1}(u, X_n^\pm) \,\}, \tag{3.64}$$

with respect to $\leq_n$. For each $n \in \mathbb{N}$ and $x \in X_n^\pm$, let $C_{n,n-1}(x) = (\pi_n^\pm)^{-1}(x)$. These sets form a partition of X_{n-1}, and satisfy

$$C_{n,n-1}(x) = \overline{C}_{n,n-1}(x) \setminus \bigcup_{x' \in X_n^\pm,\ x' <_n x} \overline{C}_{n,n-1}(x'), \tag{3.65}$$

for $x \in X_n^\pm$, by (3.60) and (3.64). For $-1 \leq m < n-1$, we continue defining sets $C_{n,m}(x)$ and $\overline{C}_{n,m}$ by reverse induction on m, taking

$$C_{n,m}(x) = \bigcup_{u \in C_{n,m+1}(x)} C_{m+1,m}(u), \quad \overline{C}_{n,m}(x) = \bigcup_{u \in \overline{C}_{n,m+1}(x)} \overline{C}_{m+1,m}(u).$$

It is straightforward to check that, for $-1 \leq l_1 < l_2 < l_3 \leq n$, we have

$$C_{l_3,l_1}(x) = \bigcup_{u \in C_{l_3,l_2}(x)} C_{l_2,l_1}(u), \quad \overline{C}_{l_3,l_1}(x) = \bigcup_{u \in \overline{C}_{l_3,l_2}(x)} \overline{C}_{l_2,l_1}(u). \tag{3.66}$$

By (3.18), (3.44) and (3.45), we have

$$\Gamma_0^\pm = R_0^\pm\,, \quad \Gamma_n^\pm = R_n^\pm \Lambda_{n-1} + \Gamma_{n-1}^\pm. \tag{3.67}$$

Lemma 3.5.1. $C_{n,-1}(x) \subset \overline{C}_{n,-1}(x) \subset B_{-1}(x, \Gamma_n^\pm)$.

Proof. We proceed by induction on n. For $n = 0$ and $x \in X_0^{\pm}$, the inclusion $C_{0,-1}(x) \subset B_{-1}(x, R_0^{\pm})$ follows from Lemma 3.4.16 and (3.65). Suppose then that we have $C_{m,-1}(y) \subset \overline{C}_{m,-1}(y) \subset B_{-1}(y, \Gamma_m^{\pm})$ for $0 \le m < n$ and $y \in X_m^{\pm}$. We have

$$C_{n,-1}(x) = \bigcup_{u \in C_{n,n-1}(x)} C_{n,n-1}(u), \quad \overline{C}_{n,-1}(x) = \bigcup_{u \in \overline{C}_{n,n-1}(x)} \overline{C}_{n,n-1}(u)$$

by (3.66). For each $u \in \overline{C}_{n,n-1}(x)$, we have $d_{n-1}(x, u) \le R_n^+$ by Lemma 3.4.16 and (3.65). Therefore $d_{-1}(x, u) \le \Lambda_{n-1} R_n^+$ by Proposition 3.4.1 (vii). Then the result follows easily from the induction hypothesis using the triangle inequality. □

Lemma 3.5.2. *For every $n \in \mathbb{N}$ and $x \in X_n^{\pm}$, we have $B_{n-1}(x, r_n^{\pm}) \subset C_{n,n-1}(x)$.*

Proof. For $u \in B_{n-1}(x, r_n^{\pm})$, we have $u \in \mathcal{Z}_{n-1}^{\pm}$ by Lemma 3.4.15 and $d_{n-1}(u, X_n) \le r_n^{\pm}$ by definition. Then the result follows from (3.64) and the fact that $X_n^{\pm}$ is $(2r_n^+ + 1)$-separated by Proposition 3.4.1 (iv). □

The following result follows from Lemma 3.5.2 by induction.

Corollary 3.5.3. *For every $n \in \mathbb{N}$ and $x \in X_n$, we have $B_{-1}(x, \sum_{i=0}^{n} r_i) \subset C_{n,n-1}(x)$.*

The following lemma states that every cluster $C_{n,n-1}(x)$ is a star-shaped subset of (X_{n-1}, E_{n-1}), with center x.

Lemma 3.5.4. *For $x \in X_n^{\pm}$ and $u \in C_{n,n-1}(x)$, any geodesic segment in (X_{n-1}, E_{n-1}) of the form $\tau = (x = \tau_0, \dots, \tau_l = u)$ is a path in $C_{n,n-1}(x)$.*

Proof. We prove that $\tau_k \in C_{n,n-1}(x)$ by reverse induction on $k = 0, \dots, l$. We have $\tau_l = u \in C_{n,n-1}(x)$ by hypothesis. Now suppose that, for some $k = 0, \dots, l-1$, we have $\tau_{k+1} \in C_{n,n-1}(x)$. Assume by absurdity that $\tau_k \notin C_{n,n-1}(x)$. Since τ is a geodesic segment, we have

$$d_{n-1}(\tau_k, X_n^{\pm}) \le d_{n-1}(\tau_k, x) = d_{n-1}(\tau_{k+1}, x) - 1 = d_{n-1}(\tau_{k+1}, X_n^{\pm}) - 1 \le d_{n-1}(\tau_k, X_n^{\pm}),$$

and therefore $\tau_k \in \overline{C}_{n,n-1}(x)$. So, according to (3.64), there must be some $y \in X_n^{\pm}$ such that $d_{n-1}(\tau_k, y) = d_{n-1}(\tau_k, x) = k$ and $y <_n x$. But then $d_{n-1}(\tau_{k+1}, y) \le k + 1 = d_{n-1}(\tau_{k+1}, x)$, yielding $\tau_{k+1} \notin C_{n,n-1}(x)$ by (3.64), a contradiction. □

Lemma 3.5.5. *Let $x \in X_n \cap B_{-1}(p, \mathfrak{r}_m - K_{n-1} - 2\Lambda_{n-1} R_n^+)$ and $(m, z) \in \mathfrak{R}_{n-1}$. Then $C_{n,n-1}(x) \subset \operatorname{dom}(\mathfrak{h}_{m,z})$ and $\mathfrak{h}_{m,z}(C_{n,n-1}(x)) = C_{n,n-1}(\mathfrak{h}_{m,z}(x))$.*

Proof. It is an immediate consequence of (3.62), (3.63), (3.65) and Lemma 3.4.4 (b). □

3.6 Colorings

3.6.1 Colorings χ_n

Given $a \in \mathbb{N}$, let $[a] = \{0, \dots, a-1\}$. For $n \in \mathbb{N}$ and $x \in X_n^\pm$, let

$$H_{n,x} = \left[\eta_n\left(\left|B_{n-1}(x, r_n^\pm)\right|\right)\right], \quad I_{n,x} = \left[5 + \left|B_{n-1}\left(x, r_n^\pm s_n\right)\right|\right]. \tag{3.68}$$

The standard ordering of $\mathbb{N}$ and the calligraphic ordering of $I_{n,x}^2$ can be used to realize $I_{n,x}^2$ as an initial segment of $\mathbb{N}$. Since $|I_{n,x}|^2 \leq |H_{n,x}|$ by Proposition 3.4.1 (iii), the sets $I_{n,x}$ and $I_{n,x}^2$ become initial segments of $H_{n,x}$. For $n \in \mathbb{N}$, let

$$\mathcal{H}_n = \bigcup_{x \in X_n} H_{n,x}, \quad \mathcal{I}_n = \bigcup_{x \in X_n} I_{n,x}. \tag{3.69}$$

Remark 3.6.1. From now on, when referring to a coloring $\phi \colon X_n \to \mathcal{H}_n$ (respectively, $\phi \colon X_n \to \mathcal{I}_n$), we assume that, for each $x \in X_n$, we have $\phi(x) \in H_{n,x}$ (respectively, $\phi(x) \in I_{n,x}$).

Proposition 3.6.1. *For every $n \in \mathbb{N}$, there is a coloring $\chi_n \colon X_n \to \mathcal{I}_n$ satisfying the following conditions:*

(i) *We have $\chi_n(x) = 0$ if and only if $x \in \mathfrak{X}_n$.*

(ii) *For all $x, y \in X_n^\pm$ with $d_{n-1}(x, y) \leq r_n^\pm s_n$, we have $\chi_n(x) < \chi_n(y)$ if and only if $x <_n y$. In particular, if $0 < d_{n-1}(x, y) \leq r_n^\pm s_n$, then $\chi_n(x) \neq \chi_n(y)$.*

(iii) *For every $(m, z) \in \mathfrak{R}_{n-1}$, the map $\mathfrak{h}_{m,z} \colon (B_n(p, \Gamma_m^+), \chi_n) \to (B_n(z, \Gamma_m^+), \chi_n)$ is color-preserving.*

Proof. First, set $\chi_n(x) = 0$ for all $x \in \mathfrak{X}_n$. Then we define $\chi_n(x)$ for $x \in X_n^\pm \setminus \mathfrak{X}_n$ by induction using $\leq_n$. Let $A_x^\pm = \{\, y \in X_n^\pm \mid y <_n x \,\}$, and let

$$\chi_n(x) = \min\{I_{n,x} \setminus (\{0\} \cup \chi_n(A_x^\pm \cap B_{n-1}(x, r_n^\pm s_n)))\}. \tag{3.70}$$

Note that this is well defined since

$$|A_x^\pm \cap B_{n-1}(x, r_n^\pm s_n)| \leq |B_{n-1}(x, r_n^\pm s_n)| - 1 \leq |I_{n,x}| - 1.$$

With this definition, it is obvious that χ_n satisfies (i) and (ii).

To prove (iii), we show by induction on $(X_n \setminus \mathfrak{X}_n, \leq_n)$ that, if $x \subset B_n(z, \Gamma_m^+)$ for $(m, z) \in \mathfrak{R}_{n-1}$, then $\chi_n(x) = \chi_n(\mathfrak{h}_{m,z}^{-1}(x))$. By Remark 3.4.4, the set $X_n \cap B_{-1}(p, \mathfrak{r}_m - K_{n-1})$ is an initial segment of $(X_n, \leq_n)$. For $x \in X_n \cap B_{-1}(p, \mathfrak{r}_m - K_{n-1})$, the result is trivial since

$\mathfrak{h}_{m,p}$ is the identity. Suppose $x \in X_n \cap B_n(z, \Gamma_m^+)$ for some $(m, z) \in \mathfrak{R}_{n-1}$ with $z \neq p$. By (3.22) and (3.45), we have $B_{n-1}(x, r_n^\pm s_n) \subset B_{-1}(z, \mathfrak{r}_m - K_{n-1})$. Thus

$$\mathfrak{h}_{m,z} \colon (B_{n-1}(\mathfrak{h}_{m,z}^{-1}(x), r_n^\pm s_n), \leq_n) \to (B_{n-1}(x, r_n^\pm s_n), \leq_n) \tag{3.71}$$

is order-preserving and an $r_n^\pm s_n$-short scale isometry w.r.t. d_{n-1} by Proposition 3.4.1 (viii) and Lemma 3.4.4 (b). Therefore $A_x^\pm \cap B_{n-1}(x, r_n^\pm s_n) = \mathfrak{h}_{m,z}(A^\pm_{\mathfrak{h}_{m,z}^{-1}(x)} \cap B_{n-1}(\mathfrak{h}_{m,z}^{-1}(x), r_n^\pm s_n))$. Then, by the induction hypothesis, we have

$$\chi_n(A_x^\pm \cap B_{n-1}(x, r_n^\pm s_n)) = \chi_n(A^\pm_{\mathfrak{h}_{m,z}^{-1}(x)} \cap B_{n-1}(\mathfrak{h}_{m,z}^{-1}(x), r_n^\pm s_n)).$$

Moreover $I_{n,x} = I_{n,\mathfrak{h}_{m,y}^{-1}(x)}$ because (3.71) is order-preserving and an $r_n^\pm s_n$-short scale isometry with respect to d_{n-1}. Then the result follows from (3.70). □

3.6.2 Equivalences

We will define, by induction on $n \in \mathbb{N}$, the notion of n-equivalence between points $x, y \in X_n$. In addition, an explicit family of n-equivalences will be constructed, together with an induced equivalence relation.

Consider the restriction of the graph structure E_{n-1} to $C_{n,n-1}(x)$, for every $n \in \mathbb{N}$ and $x \in X_n$.

Definition 3.6.2. For $x, y \in X_0$, a 0-*equivalence* is a pointed graph isomorphism

$$f \colon (\overline{C}_{0,-1}(x), x) \to (\overline{C}_{0,-1}(y), y)$$

such that $f(C_{0,-1}(x)) = C_{0,-1}(f(x))$.

Let $\sim_0^\pm$ be the equivalence relation on $X_0^\pm$ defined by declaring $x \sim_0^\pm y$ for $x, y \in X_0^\pm$ if there is some 0-equivalence $(\overline{C}_{0,-1}(x), x) \to (\overline{C}_{0,-1}(y), y)$. Let Φ_0 be the map defined on $X_0 = X_0^+ \sqcup X_0^-$ that sends each point $x \in X_0^\pm$ to its equivalence class with respect to $\sim_0^\pm$. The range of this map is obviously finite.

Lemma 3.6.3. *For $n \in \mathbb{N}$, there are disjoint subsets $X_0^{-,\Phi}, X_0^{+,\Phi} \subset X_0$ satisfying the following properties:*

(i) *The sets $X_0^{\pm,\Phi}$ are maximal among the subsets of $X_0^\pm$ where Φ_0 is injective.*

(ii) *For $u \in X_0^{\pm,\Phi}$ and $v \in X_0^\pm$, if $\Phi_0(u) = \Phi_0(v)$, then $d_0(u, p) \leq d_0(v, p)$.*

Proof. This follows by taking in each $\sim_0^\pm$-equivalence class a representative that minimizes the d_0-distance to p. □

By Lemma 3.6.3, for each point $x \in X_0^{\pm}$, there is a unique element $u \in X_0^{\pm,\Phi}$ satisfying $\Phi_0(x) = \Phi_0(u)$. Let $\operatorname{rep}_0^{\pm} \colon X_0^{\pm} \to X_0^{\pm,\Phi}$ be the maps determined by this correspondence, and let $\operatorname{rep}_0 \colon X_0 \to X_0^{\Phi} := X_0^{+,\Phi} \sqcup X_0^{-,\Phi}$ be their union.

Lemma 3.6.4. *For all $(m, y) \in \mathfrak{R}_{-1}$ and $x \in X_0^{\pm} \cap B_0(p, \Gamma_0^+)$, the following properties hold.*

(i) $\overline{C}_{0,-1}(x) \subset \operatorname{dom} \mathfrak{h}_{m,y}$.

(ii) *The restriction*

$$\mathfrak{h}_{m,y} \colon (\overline{C}_{0,-1}(x), x) \to (\overline{C}_{0,-1}(\mathfrak{h}_{m,y}(x)), \mathfrak{h}_{m,y}(x))$$

is a 0*-equivalence; in particular, $x \sim_0 \mathfrak{h}_{m,y}(x)$ and $p \sim_0 y$.*

Proof. By Lemma 3.5.1 and the triangle inequality, we have

$$\overline{C}_{0,-1}(x) \subset B_{-1}(x, \Gamma_0^+) \subset B_{-1}\left(p, \Lambda_0 \Gamma_m^+ + \Gamma_0^+\right). \tag{3.72}$$

By (3.22), (3.44) and (3.45), we have

$$\mathfrak{r}_m > 4\Lambda_m \Gamma_m^+ + K_m.$$

The assumption $(m, y) \in \mathfrak{R}_{-1}$ implies $m \geq 0$ according to (3.40). So $\Lambda_m \geq \Lambda_0 \geq \Lambda_{-1} = 1$ by (3.18) and (3.44), $K_m \geq K_0 > K_{-1} = 0$ by (3.20), (3.21), (3.44) and (3.45), and $\Gamma_m^+ \geq R_0^+$ by (3.47). Therefore

$$\mathfrak{r}_m - K_{-1} - 2\Lambda_{-1} R_0^+ > 4\Lambda_m \Gamma_m^+ + K_m - 2R_0^+ > \Lambda_0 \Gamma_m^+ + R_0^+.$$

Then (3.72) yields

$$\overline{C}_{0,-1}(x) \subset B_{-1}\left(p, \mathfrak{r}_m - K_{-1} - 2\Lambda_{-1} R_0^+\right), \tag{3.73}$$

completing the proof of (i) because $\operatorname{dom} \mathfrak{h}_{m,y} = B_{-1}(p, \mathfrak{r}_m)$.

Property (ii) follows from (3.63) and Proposition 3.4.1 (viii). □

Proposition 3.6.5. *For $x \in X_0^{\pm}$, there is a* 0*-equivalence*

$$h_{0,x} \colon \left(\overline{C}_{0,-1}\left(\operatorname{rep}_0(x)\right), \operatorname{rep}_0(x)\right) \to \left(\overline{C}_{0,-1}(x), x\right)$$

satisfying the following properties:

(i) *If $x \in X_0^{\pm,\Phi}$, then $h_{0,x}$ is the identity on $\overline{C}_{0,-1}(x)$.*

(ii) *For $(m, y) \in \mathfrak{R}_{-1}$ and $x \in X_0 \cap B_0(y, \Gamma_0^+)$, we have $h_{0,x} = \mathfrak{h}_{m,y} \circ h_{0,\mathfrak{h}_{m,y}^{-1}(x)}$.*

(iii) *If $x \in \mathfrak{X}_0$, then $h_{0,x} = \mathfrak{h}_{0,x}|_{\overline{C}_{0,-1}(x)}$.*

Proof. First, set $h_{0,x} = \mathrm{id}_{C_{0,-1}(x)}$ for every $x \in X_0^{\pm,\Phi}$, so that (i) is satisfied. Now, we define $h_{0,x}$ independently for $x \in A_m \setminus A_{m-1}$, where

$$A_m = \bigsqcup_{y \in \mathfrak{X}_m} B_n(y, \Gamma_m^+) \cap X_0 \setminus X_0^\Phi$$

for $m \geq n$, and $A_{-1} = \emptyset$. Note that A_m is a union of disjoint subsets by Proposition 3.3.4 (i), since $\mathfrak{s}_m \geq \Gamma_m^+$ by (3.23) and (3.44). This completes the definition of $h_{0,x}$ for all $x \in X_0$ because $X_0 = \bigcup_{m\geq 0} A_m$ since $p \in \mathfrak{X}_m$ (Proposition 3.3.4 (i)) and $\Gamma_m^+ \uparrow \infty$. Moreover (iii) is a direct consequence of (i) and (ii), and therefore we only have to check (ii).

Let $x \in A_m \setminus A_{m-1}$ for $m \geq 0$. On the one hand, if

$$x \in \big(B_0(p, \Gamma_m^+) \setminus X_0^\Phi\big) \setminus A_{m-1},$$

then let $h_{0,x} \colon (C_{0,-1}(\mathrm{rep}_0(x)), \mathrm{rep}_0(x)) \to (C_{0,-1}(x), x)$ be any 0-equivalence. On the other hand, if

$$x \in \big(B_0(y, \Gamma_m^+) \setminus X_0^\Phi\big) \setminus A_{m-1}$$

for some $y \in \mathfrak{X}_m \setminus \{p\}$, then $\mathrm{rep}_0(x) \in B_0(p, \Gamma_m^+)$ by Lemma 3.6.3 (ii), and let

$$h_{0,x} = \mathfrak{h}_{m,y} \circ h_{0,\mathfrak{h}_{m,y}^{-1}(x)}.$$

Note that this composite is well defined because

$$\operatorname{im} h_{0,\mathfrak{h}_{m,y}^{-1}(x)} = B_{-1}(x, r_0^\pm) \subset B_{-1}(x, R_0^\pm) \subset \operatorname{dom} \mathfrak{h}_{m,y}$$

by Lemma 3.6.4 (i) and (3.49). Property (ii) is obvious with this definition of $h_{0,x}$. □

Now, given any integer $n > 0$, suppose that we have already defined the equivalences relations $\sim_m$, the sets X_m^Φ, and maps rep_m and $h_{m,x}$ for $0 \leq m < n$. Let

$$\mathfrak{C}_{n,-1}(x) = \bigcup_{v \in B_n(x,n)} \overline{C}_{n,-1}(v), \quad \mathfrak{C}_{n,n-1}(x) = \bigcup_{v \in B_n(x,n)} \overline{C}_{n,n-1}(v)$$

Definition 3.6.6. For $n \in \mathbb{N}$ and $x, y \in X_n^\pm$, a pointed graph isomorphism

$$f \colon (\mathfrak{C}_{n,-1}(x), x) \to (\mathfrak{C}_{n,-1}(y), y)$$

is an *n-equivalence* from x to y, denoted by $f \colon x \to y$, if it satisfies the following properties for $0 \leq m < n$ and $v \in B_n(x, n)$:

(i) We have $f(B_n(x, n)) = B_n(f(x), n)$.

(ii) We have $f(\overline{C}_{n,n-1}(v)) = \overline{C}_{n,n-1}(f(v))$ and $f(C_{n,n-1}(v)) = C_{n,n-1}(f(v))$.

(iii) We have

$$f\left(X^{\pm}_{n-1} \cap \mathcal{C}_{n,n-1}(x)\right) = X^{\pm}_{n-1} \cap \mathcal{C}_{n,n-1}(y),$$

and

$$f\colon \left(X^{\pm}_{n-1} \cap \mathcal{C}_{n,n-1}(x), \chi_{n-1}\right) \to \left(X^{\pm}_{n-1} \cap \mathcal{C}_{n,n-1}(y), \chi_{n-1}\right)$$

is a color-preserving graph isomorphism with respect to E_{n-1}.

(iv) We have

$$f\left(\mathfrak{X}_{n-1} \cap \mathcal{C}_{n,n-1}(x)\right) = \mathfrak{X}_{n-1} \cap \mathcal{C}_{n,n-1}(y).$$

(v) For all $u \in \operatorname{Pen}_{n-1}(C_{n,n-1}(x), 1)$, the restriction $f\colon \mathcal{C}_{n-1,-1}(u) \to \mathcal{C}_{n-1,-1}(f(u))$ equals $h_{n-1,f(u)} \circ h^{-1}_{n-1,u}$; in particular, it is an $(n-1)$-equivalence.

Remark 3.6.2. Note that $X^{\pm}_{n-1} \cap \overline{C}_{n,n-1}(x), \overline{C}_{n-1,-1}(u) \subset \overline{C}_{n,-1}(x)$ by (3.66).

Remark 3.6.3. For every $u \in \operatorname{Pen}_{n-1}(C_{n,n-1}(x), 1)$ and $v \in B_{n-1}(u, n-1)$, we have $d_n(x, \pi_n(v)) \leq n$ by Proposition 3.4.1 (vi) and the definition of E_n. So $\overline{C}_{n-1,-1}(v) \subset \operatorname{dom} f$ in Definition 3.6.6 (v).

The following lemma is an immediate consequence of Definitions 3.6.6 and 3.6.16.

Lemma 3.6.7. *The family of n-equivalences between points of $X^{\pm}_n$ is closed by the operations of composition and inversion of maps.*

According to Lemma 3.6.7, for $n \in \mathbb{N}$, an equivalence relation $\sim^{\pm}_n$ on $X^{\pm}_n$ is defined by declaring $x \sim^{\pm}_n y$ if there is some n-equivalence between x and y. Let Φ_n be the map defined on $X_n = X^+_n \sqcup X^-_n$ that sends each point $x \in X^{\pm}_n$ to its equivalence class with respect to $\sim^{\pm}_n$. The range of each of these maps is obviously finite.

Lemma 3.6.8. *For $n \in \mathbb{N}$, there are disjoint subsets $X^{-,\Phi}_n, X^{+,\Phi}_n \subset X_n$ satisfying the following properties:*

(i) *The sets $X^{\pm,\Phi}_n$ are maximal among the subsets of $X^{\pm}_n$ where Φ_n is injective.*

(ii) *For $u \in X^{\pm,\Phi}_n$ and $v \in X^{\pm}_n$, if $\Phi_n(u) = \Phi_n(v)$, then $d_n(u,p) \leq d_n(v,p)$.*

Proof. This follows by taking in each $\sim^{\pm}_n$-equivalence class a representative that minimizes the d_n-distance to p. □

By Lemma 3.6.8, for each point $x \in X^{\pm}_n$, there is a unique element $u \in X^{\pm,\Phi}_n$ satisfying $\Phi_n(x) = \Phi_n(u)$. Let $\operatorname{rep}^{\pm}_n\colon X^{\pm}_n \to X^{\pm,\Phi}_n$ be the maps determined by this correspondence, and let $\operatorname{rep}_n\colon X_n \to X^{\Phi}_n := X^{+,\Phi}_n \sqcup X^{-,\Phi}_n$ be their union.

Lemma 3.6.9. *For all $(m, y) \in \mathfrak{R}_{n-1}$ and $x \in X^{\pm}_n \cap B_n(p, \Gamma^+_m)$, the following properties hold:*

(i) $\mathcal{C}_{n,-1}(v) \subset \operatorname{dom} \mathfrak{h}_{m,y}$.

(ii) *The restriction*

$$\mathfrak{h}_{m,y} \colon (\mathcal{C}_{n,-1}(x), x) \to (\mathcal{C}_{n,-1}(\mathfrak{h}_{m,y}(x)), \mathfrak{h}_{m,y}(x))$$

is an n-equivalence; in particular, $x \sim_n \mathfrak{h}_{m,y}(x)$ and $p \sim_n y$.

Proof. By Lemma 3.5.1, for every $v \in B_n(x,n)$, we have $\overline{C}_{n,-1}(v) \subset B_{-1}(v, \Gamma_n^+)$. Using the triangle inequality, we get

$$\mathcal{C}_{n,-1}(v) \subset B_{-1}(x, \Gamma_n^+ + n\Lambda_n) \subset B_{-1}\left(p, \Lambda_n(\Gamma_m^+ + n) + \Gamma_n^+\right). \tag{3.74}$$

By (3.22), (3.44) and (3.45), we have

$$\mathfrak{r}_m > 4\Lambda_m(\Gamma_m^+ + m) + K_m.$$

The assumption $(m,y) \in \mathfrak{R}_{n-1}$ implies $m \geq n$ according to (3.36). So $\Lambda_m \geq \Lambda_n > \Lambda_{n-1}$ by (3.18) and (3.44), $K_m \geq K_n > K_{n-1}$ by (3.20), (3.21), (3.44) and (3.45), and $\Gamma_m^+ \geq R_n^+$ by (3.47). Therefore

$$\mathfrak{r}_m - K_n > 4\Lambda_m(\Gamma_m^+ + m) + K_m - K_n > \Lambda_n(\Gamma_m^+ + n) + \Gamma_n^+.$$

Then (3.74) yields

$$\mathcal{C}_{n,-1}(v) \subset B_{-1}\left(p, \mathfrak{r}_m - K_n\right), \tag{3.75}$$

completing the proof of (i) because $\operatorname{dom} \mathfrak{h}_{m,y} = B_{-1}(p, \mathfrak{r}_m)$.

Let us prove (ii). We proceed by induction on n. For $n = 0$, the result follows from Lemma 3.6.4 (ii). So suppose that, for $n > 0$, the result is true for $0 \leq m < n$. Definition 3.6.6 (i) follows from Proposition 3.4.1 (viii) and (3.75). We get $\mathfrak{h}_{m,y}(\overline{C}_{n,n-1}(u)) = \overline{C}_{n,n-1}(\mathfrak{h}_{m,y}(u))$ and $\mathfrak{h}_{m,y}(C_{n,n-1}(u)) = C_{n,n-1}(\mathfrak{h}_{m,y}(u))$ for every $v \in B_n(x,n)$ and $u \in \overline{C}_{n,l}(v)$ using Lemma 3.5.5, (3.63) and (3.75),. Thus Definition 3.6.6 (ii) is satisfied. The map

$$\mathfrak{h}_{m,y} \colon \mathcal{C}_{n,n-1}(v) \to \mathcal{C}_{n,n-1}(w)$$

is a graph isomorphism that preserves χ_{n-1} by Propositions 3.4.1 (viii) and 3.6.1 (iii). Therefore

$$\mathfrak{h}_{m,y}(X_m^\pm \cap \mathcal{C}_{n,n-1}(x)) = X_m^\pm \cap \mathcal{C}_{n,n-1}(y)$$

by Proposition 3.4.1 (ii),(viii). Hence $\mathfrak{h}_{m,y}$ satisfies Definition 3.6.6 (iii). Then Definition 3.6.6 (v) follows by the induction hypothesis. By Proposition 3.3.4 (iii), we have $\mathfrak{X}_{n-1} \cap \mathfrak{B}_{n-1}^m(y) = \mathfrak{h}_{m,y}(\mathfrak{Z}_{n-1}^m)$ for each $(m,y) \in \mathfrak{R}_{n-1}$. In particular, for $(m,y) = (m,p)$, we obtain $\mathfrak{Z}_{n-1}^m = \mathfrak{X}_{n-1} \cap \mathfrak{B}_{n-1}^m(p)$. So $\mathfrak{X}_{n-1} \cap \mathfrak{B}_{n-1}^m(y) = \mathfrak{h}_{m,y}(\mathfrak{X}_{n-1} \cap \mathfrak{B}_{n-1}^m(p))$, and Definition 3.6.6 (iv) follows using (3.74) and (i), since $\mathfrak{r}_m \geq R_n^+ \geq r_n^\pm$ according to (3.47)–(3.49). Therefore $\mathfrak{h}_{m,y}$ satisfies Definition 3.6.6 (iv). This completes the proof of (ii). $\square$

Proposition 3.6.10. *For $n \in \mathbb{N}$ and $x \in X_n$, there is an n-equivalence $h_{n,x} \colon \operatorname{rep}_n(x) \to x$ satisfying the following properties:*

(i) *If $x \in X_n^\Phi$, then $h_{n,x}$ is the identity on $\mathcal{C}_{n,-1}(x)$.*

(ii) *For $(m, y) \in \mathfrak{R}_{n-1}$ and $x \in X_n \cap B_n(y, \Gamma_m^+)$, we have $h_{n,x} = \mathfrak{h}_{m,y} \circ h_{n,\mathfrak{h}_{m,y}^{-1}(x)}$.*

(iii) *If $x \in \mathfrak{X}_n$, then $h_{n,x} = \mathfrak{h}_{n,x}$ on $\mathcal{C}_{n,-1}(x)$.*

Proof. First, set define $h_{n,x}$ as the identity on $\mathcal{C}_{n,-1}(x)$ for every $x \in X_n^\Phi$, so that (i) is satisfied. Now, we define $h_{n,x}$ independently for $x \in A_m \setminus A_{m-1}$, where

$$A_m = \bigsqcup_{y \in \mathfrak{X}_m} B_n(y, \Gamma_m^+) \cap X_n \setminus X_n^\Phi$$

for $m \geq n$, and $A_{n-1} = \emptyset$. Note that A_m is a union of disjoint subsets by Proposition 3.3.4 (i), since $\mathfrak{s}_m \geq \Gamma_m^+$ by (3.23) and (3.44). This completes the definition of $h_{n,x}$ for all $x \in X_n$ because $X_n = \bigcup_{m \geq n} A_m$ since $p \in \mathfrak{X}_m$ (Proposition 3.3.4 (i)) and $\Gamma_m^+ \uparrow \infty$. Moreover (iii) is a direct consequence of (i) and (ii), and therefore we only have to check (ii).

Let $x \in A_m \setminus A_{m-1}$ for $m \geq n$. On the one hand, if

$$x \in \left(B_n(p, \Gamma_m^+) \cap X_n \setminus X_n^\Phi\right) \setminus A_{m-1},$$

then let $h_{n,x} \colon \operatorname{rep}_n(x) \to x$ be any n-equivalence, whose existence is guaranteed by the definition of rep_n. On the other hand, if

$$x \in \left(B_n(y, \Gamma_m^+) \cap X_n \setminus X_n^\Phi\right) \setminus A_{m-1}$$

for some $y \in \mathfrak{X}_m \setminus \{p\}$, then $\operatorname{rep}_n(x) \in B_n(p, \Gamma_m^+)$ by Lemmas 3.6.3 (ii) and 3.6.8 (ii), and let $h_{n,x} = \mathfrak{h}_{m,y} \circ h_{n,\mathfrak{h}_{m,y}^{-1}(x)}$. Note that this composite is well defined because, for $x \in X_n^\pm$,

$$\operatorname{im} h_{n,\mathfrak{h}_{m,y}^{-1}(x)} = B_{n-1}(x, r_n^\pm) \subset B_{n-1}(x, R_n^\pm) \subset \operatorname{dom} \mathfrak{h}_{m,y}$$

by Lemma 3.6.9 (i) and (3.49). Property (ii) is obvious with this definition of $h_{n,x}$. □

Remark 3.6.4. In accordance with the discussion at the beginning of Section 3.4, only Proposition 3.6.10 (i) is needed to prove Theorem 3.1.1, whereas the whole Proposition 3.6.10 is needed to prove Theorem 3.1.2.

Remark 3.6.5. Note that the definitions of $\sim_n^\pm$, Φ_n e $\operatorname{rep}_n^\pm$, and the properties of $X_n^{\pm,\Phi}$ already guarantee the existence of n-equivalences $h_{n,x}$. Moreover there is no problem to assume (i) and (iii). So the really new contribution of Proposition 3.6.10 is (ii).

3.6.3 Weak equivalences

Definition 3.6.11. For $x, y \in X_0$, a 0-*weak equivalence* from x to y, denoted $f\colon x \to y$, is a pointed graph isomorphism $(B_{-1}(x, r_n^\pm), x) \to (B_{-1}(y, r_n^\pm), y)$

Let $\widehat{\sim}_0^\pm$ be the equivalence relation on $X_0^\pm$ defined by declaring $x \widehat{\sim}_0^\pm y$ for $x, y \in X_0^\pm$ if there is some 0-weak equivalence $(B_{-1}(x, r_0^\pm), x) \to (B_{-1}(y, r_0^\pm), y)$. Let $\widehat{\Phi}_0$ be the map defined on $X_0 = X_0^+ \sqcup X_0^-$ that sends each point $x \in X_0^\pm$ to its equivalence class with respect to $\widehat{\sim}_0^\pm$. The range of this map is obviously finite.

Lemma 3.6.12. *Let $f\colon x \to y$ be a 0-equivalence. Then the restriction of f to $B_{-1}(x, r_0^\pm)$ is a 0-weak equivalence; in particular, $x \sim_0 y$ implies $x \widehat{\sim}_0 y$.*

Lemma 3.6.13. *For $n \in \mathbb{N}$, there are disjoint subsets $X_0^{-,\widehat{\Phi}}, X_0^{+,\widehat{\Phi}} \subset X_0$ satisfying the following properties:*

(i) *The sets $X_0^{\pm,\widehat{\Phi}}$ are maximal among the subsets of $X_0^\pm$ where $\widehat{\Phi}_0$ is injective.*

(ii) *For $u \in X_0^{\pm,\widehat{\Phi}}$ and $v \in X_0^\pm$, if $\widehat{\Phi}_0(u) = \widehat{\Phi}_0(v)$, then $d_0(u, p) \le d_0(v, p)$.*

(iii) *We have $X_0^{\pm,\widehat{\Phi}} \subset X_0^{\pm,\Phi}$.*

Proof. This follows by taking in each $\widehat{\sim}_0^\pm$-equivalence class a representative that minimizes the d_0-distance to p. □

By Lemma 3.6.3, for each point $x \in X_0^\pm$, there is a unique element $u \in X_0^{\pm,\widehat{\Phi}}$ satisfying $\widehat{\Phi}_0(x) = \widehat{\Phi}_0(u)$. Let $\widehat{\operatorname{rep}}_0^\pm\colon X_0^\pm \to X_0^{\pm,\widehat{\Phi}}$ be the maps determined by this correspondence, and let $\widehat{\operatorname{rep}}_0\colon X_0 \to X_0^{\widehat{\Phi}} := X_0^{+,\widehat{\Phi}} \sqcup X_0^{-,\widehat{\Phi}}$ be their union.

The following lemma follows from Lemmas 3.6.4 and 3.6.12.

Lemma 3.6.14. *For all $(m, y) \in \mathfrak{R}_{-1}$ and $x \in X_0^\pm \cap B_0(p, \Gamma_0^+)$, the following properties hold.*

(i) $B_{-1}(x, r_0^\pm)(x) \subset \operatorname{dom} \mathfrak{h}_{m,y}$.

(ii) *The restriction*

$$\mathfrak{h}_{m,y}\colon (B_{-1}(x, r_0^\pm)(x), x) \to (B_{-1}(x, r_0^\pm)(x), \mathfrak{h}_{m,y}(x))$$

is a 0-weak equivalence; in particular, $x \,\widehat{\sim}_0\, \mathfrak{h}_{m,y}(x)$ and $p \,\widehat{\sim}_0\, y$.

Proposition 3.6.15. *For $x \in X_0^\pm$, there is a 0-weak equivalence*

$$\hat{h}_{0,x}\colon \big(B_{-1}(\widehat{\operatorname{rep}}_0(x), r_0^\pm), \widehat{\operatorname{rep}}_0(x)\big) \to \big(B_{-1}(x, r_0^\pm), x\big)$$

satisfying the following properties:

(i) *If* $x \in X_0^{\pm,\widehat{\Phi}}$, *then* $\hat{h}_{0,x}$ *is the identity on* $B_{-1}(x, r_0^{\pm})(x)$.

(ii) *For all* $x \in X_0^{\pm}$, $\hat{h}_{0,x} = h_{0,x} \circ \hat{h}_{0,\mathrm{rep}_0(x)}$.

Proof. First, for every $x \in X_0^{\pm,\widehat{\Phi}}$, let $\hat{h}_{0,x}$ be the identity on $B_{-1}(x, r_0^{\pm})$. Then, for points $x \in X_0^{\pm,\Phi} \setminus X_0^{\pm,\widehat{\Phi}}$, let $\hat{h}_{0,x} \colon \widehat{\mathrm{rep}}_0(x) \to x$ be any 0-weak equivalence. Finally, for every $x \in X_0 \setminus X_0^{\pm,\Phi}$, let $\hat{h}_{0,x} = h_{0,x} \circ \hat{h}_{0,\mathrm{rep}_0(x)}$. □

Now, given any integer $n > 0$, suppose that we have already defined the equivalences relations $\widehat{\sim}_m$, the sets $X_m^{\widehat{\Phi}}$, and maps $\widehat{\mathrm{rep}}_m$ and $\hat{h}_{m,x}$ for $0 \leq m < n$. Let

$$\mathscr{C}_n(x) = \bigcup_{u \in B_{n-1}(x, r_n^{\pm})} \overline{C}_{n-1,-1}(u).$$

Definition 3.6.16. For $n \in \mathbb{N}$ and $x, y \in X_n^{\pm}$, a pointed graph isomorphism

$$f \colon (\mathscr{C}_n(x), x) \to (\mathscr{C}_n(y), y)$$

is an *n-weak equivalence* from x to y, denoted $f : x \to y$, if it satisfies the following properties for $0 \leq m < n$ and $v \in B_n(x, n)$:

(i) We have $f(B_{n-1}(x, r_n^{\pm})) = B_{n-1}(y, r_n^{\pm})$.

(ii) We have

$$f\left(X_{n-1}^{\pm} \cap B_{n-1}(x, r_n^{\pm})\right) = X_{n-1}^{\pm} \cap B_{n-1}(y, r_n^{\pm}),$$

and

$$f \colon \left(X_{n-1}^{\pm} \cap B_{n-1}(x, r_n^{\pm}), \chi_{n-1}\right) \to \left(X_{n-1}^{\pm} \cap B_{n-1}(y, r_n^{\pm}), \chi_{n-1}\right)$$

is a color-preserving graph isomorphism with respect to E_{n-1}.

(iii) We have

$$f\left(\mathfrak{X}_{n-1} \cap B_{n-1}(x, r_n^{\pm})\right) = \mathfrak{X}_{n-1} \cap B_{n-1}(x, r_n^{\pm}),$$

.

(iv) For all $u \in B_{n-1}(x, r_n^{\pm} - 1)$, the restriction $f \colon \mathcal{C}_{n-1}(u) \to \mathcal{C}_{n-1}(f(u))$ equals $h_{n-1,f(u)} \circ h_{n-1,u}^{-1}$; in particular, it is an $(n-1)$-equivalence.

Remark 3.6.6. Note that for $n > 0$, $x \in X_n$ and $u \in B_{n-1}(x, r_n^{\pm} - 1)$, we have $\mathcal{C}_{n-1}(u) \subset \mathscr{C}_n(x)$ since $B_{n-1}(u, 1) \subset B_{n-1}(x, r_n^{\pm})$.

The following lemma is an immediate consequence of Definitions 3.6.6 and 3.6.16.

Lemma 3.6.17. *The family of n-weak equivalences between points of $X_n^\pm$ is closed by the operations of composition and inversion of maps. Moreover, the composition of an n-weak equivalence and an n-equivalence is an n-weak equivalence; in particular, every n-equivalence is an n-weak equivalence.*

According to Lemma 3.6.17, for $n \in \mathbb{N}$, an equivalence relation $\widehat{\sim}_n^\pm$ on $X_n^\pm$ is defined by declaring $x \widehat{\sim}_n^\pm y$ if there is some n-equivalence between x and y. Let $\widehat{\Phi}_n$ be the map defined on $X_n = X_n^+ \sqcup X_n^-$ that sends each point $x \in X_n^\pm$ to its equivalence class with respect to $\widehat{\sim}_n^\pm$. The range of each of these maps is obviously finite.

Lemma 3.6.18. *For $n \in \mathbb{N}$, there are disjoint subsets $X_n^{-,\widehat{\Phi}}, X_n^{+,\widehat{\Phi}} \subset X_n$ satisfying the following properties:*

(i) *We have $X_n^{\pm,\widehat{\Phi}} \subset X_n^{\pm,\Phi}$.*

(ii) *The sets $X_n^{\pm,\widehat{\Phi}}$ are maximal among the subsets of $X_n^\pm$ where $\widehat{\Phi}_n$ is injective.*

(iii) *For $u \in X_n^{\pm,\widehat{\Phi}}$ and $v \in X_n^\pm$, if $\widehat{\Phi}_n(u) = \widehat{\Phi}_n(v)$, then $d_n(u,p) \le d_n(v,p)$.*

Proof. This follows by taking in each $\widehat{\sim}_n^\pm$-equivalence class a representative that minimizes the d_n-distance to p. □

By Lemma 3.6.8, for each point $x \in X_n^\pm$, there is a unique element $u \in X_n^{\pm,\widehat{\Phi}}$ satisfying $\widehat{\Phi}_n(x) = \widehat{\Phi}_n(u)$. Let $\widehat{\operatorname{rep}}_n^\pm \colon X_n^\pm \to X_n^{\pm,\widehat{\Phi}}$ be the maps determined by this correspondence, and let $\widehat{\operatorname{rep}}_n \colon X_n \to X_n^{\widehat{\Phi}} := X_n^{+,\widehat{\Phi}} \sqcup X_n^{-,\widehat{\Phi}}$ be their union.

The following result follows from Lemmas 3.6.14 and 3.6.17.

Lemma 3.6.19. *For all $(m,y) \in \mathfrak{R}_{-1}$ and $x \in X_n^\pm \cap B_n(p,\Gamma_0^+)$, the following properties hold.*

(i) *$\mathscr{C}_n(x) \subset \operatorname{dom} \mathfrak{h}_{m,y}$.*

(ii) *The restriction $\mathfrak{h}_{m,y} \colon (\mathscr{C}_n(x),x) \to (\mathscr{C}_{n-1,-1}(\mathfrak{h}_{m,y}(x)),\mathfrak{h}_{m,y}(x))$ is a n-weak equivalence; in particular, $x \,\widehat{\sim}_n\, \mathfrak{h}_{m,y}(x)$ and $p \,\widehat{\sim}_n\, y$.*

Proposition 3.6.20. *For $x \in X_0^\pm$, there is a n-weak equivalence $f \colon \widehat{\operatorname{rep}}_n(x) \to x$ satisfying the following properties:*

(i) *If $x \in X_n^{\pm,\widehat{\Phi}}$, then $\hat{h}_{n,x}$ is the identity on $B_{n-1}(x, r_n^\pm)$.*

(ii) *For all $x \in X_n^\pm$, $\hat{h}_{n,x} = h_{n,x} \circ \hat{h}_{n,\operatorname{rep}_n(x)}$.*

Proof. The proof is identical to that of Proposition 3.6.15. □

3.6.4 BFS-orderings

Definition 3.6.21. Let (A, x) be a pointed connected graph with finite vertex degrees endowed with an order relation $\leq$. Define the *parent map*, $\mathrm{Pa}\colon A \setminus \{x\} \to A$, by

$$\mathrm{Pa}(u) = \min S(u, 1). \tag{3.76}$$

For $v \in A$, its *children set*, denoted by $\mathrm{Ch}(v)$, is

$$\mathrm{Ch}(v) = \mathrm{Pa}^{-1}(v) = S(v, 1) \setminus \left(\bigcup_{w<v} S(w, 1) \cup \{x\} \right). \tag{3.77}$$

Definition 3.6.22. A *BFS-ordering* on a pointed connected graph (A, x) is an order $\trianglelefteq$ on A satisfying the following conditions for all $u, v \in A$:

(i) If $d(x, u) < d(x, v)$, then $u \triangleleft v$.

(ii) If $u, v \neq x$ and $\mathrm{Pa}(u) \triangleleft \mathrm{Pa}(v)$, then $u \triangleleft v$.

The acronym "BSF" stands for "breadth-first search", which is a graph algorithm that uses this type of orderings. There exists a BFS-ordering $\trianglelefteq$ on any pointed connected graph (A, x) with finite vertex degrees. It can be defined on $B(x, n)$ by induction on $n \in \mathbb{N}$ as follows. First, declare x to be the least element in A. Then the restriction of $\trianglelefteq$ to $S(x, 1)$ is any order, and declare the points in $B(x, 1)$ to be an initial segment of $\trianglelefteq$. Next, the restriction of $\trianglelefteq$ to $S(x, 2)$ is any order such that $u \triangleleft v$ if

$$\min(S(1, u) \cap B(1, x)) \triangleleft \min(S(1, v) \cap B(1, x)),$$

and so on. This argument gives the following result.

Lemma 3.6.23. *Let $a \in \mathbb{N}$, let (A, x) be a pointed connected graph with finite vertex degrees. Then there is a BFS-ordering $\trianglelefteq$ on (A, x).*

Given an isomorphism of graphs, $f\colon A \to B$, and an order relation $\leq_A$ on A ($\leq_A \subset A \times A$), the corresponding push-forward order relation on B is $(f \times f)(\leq_A) \subset B \times B$, simply denoted by $f(\leq_A)$.

Recall that $C_{n,n-1}(x)$ is a connected subgraph of (X_{n-1}, E_{n-1}) by Lemma 3.5.4. Consider the n-equivalences $h_{n,x}$, for $n \in \mathbb{N}$ and $x \in X_n$, given by Proposition 3.6.10.

Proposition 3.6.24. *For $n \in \mathbb{N}$ and $x \in X_n$, there is a BFS-ordering $\trianglelefteq_{n,x}$ on the pointed connected graph $(C_{n,n-1}(x), x)$ satisfying $\trianglelefteq_{n,x} = h_{n,\mathrm{rep}_n(x)}(\trianglelefteq_{n,\mathrm{rep}_n(x)})$.*

Proof. Take any BFS-ordering $\trianglelefteq_{n,x}$ on $(C_{n,n-1}(x), x)$ for $x \in X_n^\Phi$ (Lemma 3.6.23). Then define $\trianglelefteq_{n,x} = h_{n,\mathrm{rep}_n(x)}(\trianglelefteq_{n,\mathrm{rep}_n(x)})$ for $x \in X_n \setminus X_n^\Phi$. □

From now on, for every $n \in \mathbb{N}$ and $x \in X_n$, the notation $\mathrm{Pa}_{n,x}$ and $\mathrm{Ch}_{n,x}$ is used for the parent map and children sets on the pointed connected graph $(C_{n,n-1}(x), x)$, with the BFS-ordering $\trianglelefteq_{n,x}$ given by Proposition 3.6.24.

Lemma 3.6.25. *Let $n \in \mathbb{N}$ and $x \in X_n$. The following properties hold for every $u \in C_{n,n-1}(x)$:*

(i) *If $u \neq x$, then $d_{n-1}(x, \mathrm{Pa}_{n,x}(u)) = d_{n-1}(x,u) - 1$.*

(ii) *We have*
$$\bigsqcup_{v \in C_{n,n-1}(x)} \mathrm{Ch}_{n,x}(v) = C_{n,n-1}(x) \setminus \{x\}.$$

(iii) *If $u \neq x$, then $|\mathrm{Ch}_{n,x}(u)| \leq \Delta_{n-1} - 1$.*

Proof. Property (i) is an easy consequence of Definitions 3.6.21 and 3.6.22 (i). Property (iii) follows from (i) and Definition 3.6.21, whereas (ii) is obvious. □

3.6.5 Colorings $\phi^i_{0,x}$

Definition 3.6.26. For $x \in X_0$, a coloring $\phi \colon C_{0,-1}(x) \to [\Delta]$ is said to be *adapted* if it satisfies the following two conditions:

(i) There is a geodesic segment in (X_{-1}, E_{-1}) of the form $\tau = (x = \tau_0, \dots, \tau_5)$ such that
$$\phi^{-1}(0) \cap B_{-1}(x,7) = \begin{cases} \{\tau_0, \tau_1, \tau_2, \tau_5\} & \text{if } x \in X_0^- \\ \{\tau_0, \tau_1, \tau_2, \tau_4, \tau_5\} & \text{if } x \in X_0^+. \end{cases}$$

(ii) For all $u \in C_{0,-1}(x)$, the coloring ϕ is injective on $\mathrm{Ch}_{0,x}(u)$.

It is said that ϕ is *strongly adapted* if it is adapted and moreover the following property holds:

(iii) We have $\phi^{-1}(0) \setminus B_{-1}(x,7) = \emptyset$.

Lemma 3.6.27. *For every $x \in X_0^\pm$, there is a strongly adapted coloring $\phi_x \colon C_{0,-1}(x) \to [\Delta]$.*

Proof. First, choose a geodesic segment in (X_{-1}, E_{-1}) of the form $\tau = (x = \tau_0, \dots, \tau_5)$, which is contained in $C_{0,-1}(x)$ because $B_{-1}(x, r_0^\pm) \subset C_{0,-1}(x)$ (Lemma 3.5.2), and $r_0^\pm > 2^{11}$ by (3.43) and (3.48). Consider the set $T_0^- = \{\tau_0, \tau_1, \tau_2, \tau_5\}$ if $x \in X_0^-$, or $T_0^+ = \{\tau_0, \tau_1, \tau_2, \tau_4, \tau_5\}$ if $x \in X_0^+$. Color $T_0^\pm$ with the color 0. In particular $\phi_x(x) = 0$. The sets $\mathrm{Ch}_{0,x}(u)$, for $u \in C_{0,-1}(x)$, form a partition of $C_{0,-1}(x) \setminus \{x\}$ by Lemma 3.6.25 (ii). Moreover $|\mathrm{Ch}_{0,x}(u) \setminus T_n^\pm| \leq \Delta - 1$ by Lemma 3.6.25 (iii). Therefore, for each $u \in$

$C_{0,-1}(x)$, we can color the points in $\mathrm{Ch}_{0,x}(u) \setminus T_n^{\pm}$ with different colors from $\{1,\dots,\Delta-1\}$. This procedure defines a coloring of $\phi_x\colon C_{0,-1}(x) \to [\Delta]$ satisfying all conditions of Definition 3.6.26. □

For colored graphs, (X,ϕ) and (Y,ψ), and a graph isomorphism, $h : X \to Y$, the notation $h(\phi) = \psi$ means $h^*\psi = \phi$.

Proposition 3.6.28. *There is a family of strongly adapted colorings, $\phi^0_{0,x}\colon C_{0,-1}(x) \to [\Delta]$, for $x \in X_0$, satisfying $\phi^0_{0,x} = h_{0,x}(\phi^0_{0,\mathrm{rep}_0(x)})$.*

Proof. If $x \in X_0^{\Phi}$, take any strongly adapted coloring (Lemma 3.6.27). If $x \in X_0 \setminus X_0^{\Phi}$, let $\phi^0_{0,x} = h_{0,x}(\phi^0_{0,\mathrm{rep}_0(x)})$. □

Proposition 3.6.29. *There is a family of colorings, $\phi^i_{0,x}\colon C_{0,-1}(x) \to [\Delta]$, for $x \in X_0$ and $i \in H_{0,x}$, satisfying the following properties:*

(i) *The coloring $\phi^0_{0,x}$ is strongly adapted.*

(ii) *We have $\phi^i_{0,x} = h_{0,x}(\phi^i_{0,\mathrm{rep}_0(x)})$.*

(iii) *For $i \in H_{0,x}$, the coloring $\phi^i_{0,x}$ is adapted.*

(iv) *For $x \in X_0$ and $i,j \in H_{0,x}$, let $A = C_{0,-1}(x)$ (respectively, $A = B_{-1}(x, r_n^{\pm})$), and let $f\colon (A, x, \phi^i_{0,x}) \to (A, x, \phi^j_{0,x})$ be a color-preserving 0-equivalence (respectively, 0-weak equivalence). Then f is the identity map on A, and $i = j$.*

Proof. First, for $i = 0$, we take the strongly adapted colorings $\phi^0_{0,x}$ constructed in Proposition 3.6.28. So (i) is satisfied.

For every $x \in X_0^{\pm,\Phi}$, choose a maximal 3-separated subset $N_{0,x}$ of $C_{-1}(x, 10, r_0^{\pm})$, together with an enumeration of its powerset,

$$\mathcal{P}(N_{0,x}) = \{\, \mathcal{N}^0_{0,x} = \emptyset,\ \mathcal{N}^1_{0,x},\ \dots \}.$$

We have $|B_{-1}(x,10)| \le \Delta^{11}$ by (2.5). Therefore $|C_{-1}(x,10,r_0^{\pm})| \ge |B_{-1}(x,r_n^{\pm})| - \Delta^{11}$ (recall that $r_0^{\pm} > 2^{11}$). By Lemma 2.1.5, $N_{0,x}$ is a 2-net in $C_{-1}(x,10,r_0^{\pm})$. So

$$|N_{0,x}| \ge \left\lfloor \frac{|B_{-1}(x,r_n^{\pm})| - \Delta^{11}}{\Delta^3} \right\rfloor \tag{3.78}$$

by Lemma 2.2.13. Therefore

$$|\mathcal{P}(N_{0,x})| \ge \exp_2 \left\lfloor \frac{|B_{-1}(x,r_n^{\pm})| - \Delta^{11}}{\Delta^3} \right\rfloor = \eta_0(|B_{-1}(x,r_n^{\pm})|).$$

Thus an injective map $H_{0,x} \to \mathcal{P}(N_{0,x})$ is well defined by $i \mapsto \mathcal{N}^i_{0,x}$.

If $x \notin X_0^\Phi$, let $N_{0,x} = h_{0,x}(N_{0,\mathrm{rep}_0(x)})$ and $\mathcal{N}^i_{0,x} = h_{0,x}(\mathcal{N}^i_{0,\mathrm{rep}_0(x)})$, and $N_{0,x}$ also satisfies (3.78). Then define

$$\phi^i_{0,x}(u) = \begin{cases} \phi^0_{0,x}(u) & \text{if } u \notin \mathcal{N}^i_{0,x} \\ 0 & \text{if } u \in \mathcal{N}^i_{0,x}. \end{cases}$$

Note that this definition agrees with the previous one in the case $i = 0$. Property (ii) follows immediately from Proposition 3.6.28 and the fact that $\mathcal{N}^i_{0,x} = h_{0,x}(\mathcal{N}^i_{0,\mathrm{rep}_0(x)})$.

To prove (iii), note that $\phi^i_{0,x} = \phi^0_{0,x}$ on $B_{-1}(x, 10)$ by construction. So Definition 3.6.26 (i) is trivially satisfied by $\phi^i_{0,x}$. For every $u \in C_{0,-1}(x)$, we have $\mathrm{Ch}_{0,x}(u) \subset B_{-1}(u, 1)$, which yields $d(v, w) \leq 2$ for all $v, w \in \mathrm{Ch}_{0,x}(u)$. Hence $N_{0,x} \cap \mathrm{Ch}_{0,x}(u)$ has at most one point because $N_{0,x}$ is 3-separated, and therefore $\mathcal{N}^i_{0,x} \cap \mathrm{Ch}_{0,x}(u)$ has at most one point. The coloring $\phi^0_{0,x}$ assigns different colors to all points in $\mathrm{Ch}_{0,x}(u)$ (Definition 3.6.26 (ii)). If $u \in B_{-1}(x, 9)$, then $\mathrm{Ch}_{0,x}(u) \subset B_{-1}(x, 10)$, and therefore $\phi^i_{0,x}$ also assigns different colors to all points in $\mathrm{Ch}_{0,x}(u)$, since $\phi^i_{0,x} = \phi^0_{0,x}$ on $B_{-1}(x, 10)$. If $u \in C_{0,-1}(x) \setminus B_{-1}(x, 9)$, then $\phi^0_{0,x}$ assigns different colors to all points in $\mathrm{Ch}_{0,x}(u)$, all of them different from 0, and it follows from the definition that $\phi^i_{0,x}$ assigns different colors to those points too. Thus Definition 3.6.26 (ii) is satisfied by $\phi^i_{0,x}$, and the coloring $\phi^i_{0,x}$ is adapted.

To prove (iv), suppose first that $A = C_{0,-1}(x)$ and f is a 0-equivalence. For all $u \in C_{0,-1}(x)$, we show that f is the identity map on $\mathrm{Ch}_{n,x}(u)$, and that $\mathcal{N}^i_{0,x} \cap \mathrm{Ch}_{n,x}(u) = \mathcal{N}^j_{0,x} \cap \mathrm{Ch}_{n,x}(u)$, using induction on u with $\trianglelefteq_{0,x}$. This will complete the proof because it follows that f is the identity map and $\mathcal{N}^i_{0,x} = \mathcal{N}^j_{0,x}$, yielding $i = j$.

First, we have $f(x) = x$ by Definition 3.6.26 (i), since x is the unique point having the correct coloring pattern on some geodesic segment of the form $\tau = (x = \tau_0, \dots, \tau_5)$. Also, $\mathcal{N}^i_{0,x} \cap \mathrm{Ch}_{n,x}(x) = \mathcal{N}^j_{0,x} \cap \mathrm{Ch}_{n,x}(x) = \emptyset$ because $N_{0,x} \cap B(x, 10) = \emptyset$.

Suppose now that, for some $u \in C_{0,-1}(x)$ with $d_{-1}(u, x) > 0$, f is the identity map on $\mathrm{Ch}_{0,x}(v)$ and $\mathcal{N}^i_{0,x} \cap \mathrm{Ch}_{n,x}(v) = \mathcal{N}^j_{0,x} \cap \mathrm{Ch}_{n,x}(v)$ for all $v \lhd_{0,x} u$. In particular, f is the identity map on $\mathrm{Ch}_{n,x}(\mathrm{Pa}_{n,x}(u))$, and therefore $f(u) = u$. Furthermore this implies $f(\mathrm{Ch}_{0,x}(u)) = \mathrm{Ch}_{0,x}(u)$ by (3.77). By definition, for $l = i, j$, we have $\phi^l_{0,x} = \phi^0_{0,x}$ on $\mathrm{Ch}_{0,x}(u) \setminus N_{0,x}$, and $\phi^l_{0,x}(u) = 0$ if $u \in \mathcal{N}^l_{0,x}$. Recall that $N_{0,x} \cap \mathrm{Ch}_{0,x}(u)$ has at most one point, which is denoted by w. By (iii) and Definition 3.6.26 (ii), $\phi^0_{0,x}$ is injective on $\mathrm{Ch}_{0,x}(u) \setminus \{w\}$. Thus $\phi^i_{0,x}$ and $\phi^j_{0,x}$ agree and are injective on $\mathrm{Ch}_{0,x}(u) \setminus \{w\}$, and therefore f is the identity on $\mathrm{Ch}_{0,x}(u) \setminus \{w\}$. But this yields $f(w) = w$, and f is color preserving only if $\mathrm{Ch}_{0,x}(u) \cap \mathcal{N}^i_{0,x} = \mathrm{Ch}_{0,x}(u) \cap \mathcal{N}^j_{0,x}$.

The proof of (iv) when $A = B_{-1}(x, r_0^\pm)$ and f is a 0-weak equivalence is similar. □

Corollary 3.6.30. *Let $x, y \in X_0$, $i \in H_{0,x}$ and $j \in H_{0,y}$, let $A = C_{0,-1}(x)$ (respectively, $A = B_{-1}(x, r_n^{\pm})$), and let $f\colon (A, x, \phi_{0,x}^i) \to (A, x, \phi_{0,x}^j)$ be a color-preserving 0-equivalence (respectively, a 0-weak equivalence). Then $i = j$ and $f = h_{n,y} \circ h_{n,x}^{-1}$ on A.*

Proof. Suppose that $A = C_{0,-1}(x)$. Since there is a 0-equivalence between x and y, we have $\Phi_0(x) = \Phi_0(y)$ and $\operatorname{rep}_0(x) = \operatorname{rep}_0(y) =: z$. So $h_{0,x}^* \phi_{0,x}^l = \phi_{0,z}^l$ for $l = i, j$ by Proposition 3.6.29 (ii). Then

$$h_{0,y}^{-1} \circ f \circ h_{0,x}\colon (C_{0,-1}(z), z, \phi_{0,z}^i) \to (C_{0,-1}(z), z, \phi_{0,z}^j)$$

is a color-preserving 0-equivalence. The result follows from Proposition 3.6.29 (iv).

The case where $A = B_{-1}(x, r_n^{\pm})$ follows similarly. □

3.6.6 Colorings $\phi_{n,x}^i$

Definition 3.6.31. Let $x \in X_n$. A coloring $\phi\colon C_{n,n-1}(x) \to \mathfrak{I}_{n-1}$ is said to be *adapted* if the following conditions are satisfied:

(i) We have $\phi^{-1}(0) = \mathfrak{X}_{n-1} \cap C_{n,n-1}(x)$;

(ii) We have

$$\phi^{-1}(1) = \begin{cases} \{x\} & \text{if} \quad x \in X_n^- \setminus \mathfrak{X}_{n-1} \\ \emptyset & \text{otherwise.} \end{cases}$$

(iii) We have

$$\phi^{-1}(2) = \begin{cases} \{x\} & \text{if} \quad x \in X_n^+ \setminus \mathfrak{X}_{n-1} \\ \emptyset & \text{otherwise.} \end{cases}$$

(iv) If $x \in \mathfrak{X}_{n-1} \cap X_n^+$, then $\phi^{-1}(3) = \{y\}$ for some $y \in S_{n-1}(x, 1)$, otherwise $\phi^{-1}(3) = \emptyset$.

(v) If $x \in \mathfrak{X}_{n-1} \cap X_n^-$, then $\phi^{-1}(4) = \{y\}$ for some $y \in S_{n-1}(x, 1)$, otherwise $\phi^{-1}(4) = \emptyset$.

The coloring ϕ is *strongly adapted* if it is adapted and, additionally, it satisfies the following condition:

(vi) $\phi^{-1}(5) = \emptyset$.

Recall that the sets $C_{n,n-1}(x)$, for $x \in X_n$, form a partition of X_{n-1} by definition.

Lemma 3.6.32. *Consider a family of adapted colorings, $\phi_x\colon C_{n,n-1}(x) \to \mathfrak{I}_{n-1}$, for $x \in X_n$, whose combination is denoted by ϕ. For every $u \in X_{n-1}$, we have $u \in X_n$ if and only if, either $\phi(u) \in \{1, 2\}$, or $\phi(u) = 0$ and there is some $v \in S_{n-1}(u, 1)$ such that $\phi(v) \in \{3, 4\}$.*

By Proposition 3.4.1 (vi), and Lemmas 3.4.16 and 3.5.1 , we have $d_{-1}(u,v) \leq 2\Lambda_{n-1}R_n^+$ for any $u,v \in C_{n,n-1}(x)$. On the other hand, if $u,v \in \mathfrak{X}_{n-1}$, then $d_{-1}(u,v) \geq \mathfrak{s}_{n-1}$ by Proposition 3.3.4 (i). Since $\mathfrak{s}_{n-1} > 3\Lambda_{n-1}\Gamma_n^+ \geq 3\Lambda_{n-1}R_n^+$ by (3.23), (3.44) and (3.47), it follows that

$$|C_{n,n-1}(x) \cap \mathfrak{X}_{n-1}| \leq 1. \tag{3.79}$$

Lemma 3.6.33. *For every $x \in X_n$, there is a strongly adapted coloring $\phi_x \colon C_{n,n-1}(x) \to \mathfrak{I}_{n-1}$.*

Proof. First, note that $[7] \subset I_{n-1,u}$ for all $u \in C_{n,n-1}(x)$ by (3.68). Define $\phi_x(u) = 0$ for every $u \in C_{n,n-1}(x) \cap \mathfrak{X}_{n-1}$. In the case where $x \in \mathfrak{X}_{n-1}$, choose some $y \in S_{n-1}(x,1)$ and define

$$\phi_x(y) = \begin{cases} 3 & \text{if } x \in \mathfrak{X}_n^- \\ 4 & \text{if } x \notin \mathfrak{X}_n^+. \end{cases}$$

If $x \notin \mathfrak{X}_{n-1}$, set

$$\phi_x(x) = \begin{cases} 1 & \text{if } x \in \mathfrak{X}_n^- \\ 2 & \text{if } x \notin \mathfrak{X}_n^+. \end{cases}$$

Let A be the set of points in $C_{n,n-1}(x)$ that have been already colored at this point. For $u \in C_{n,n-1}(x) \setminus A$, let $\phi_x(u)$ be any color in $I_{n-1,u} \setminus [6]$. □

Proposition 3.6.34. *There is a family of strongly adapted colorings, $\phi_{n,x}^0 \colon C_{n,n-1}(x) \to \mathfrak{I}_{n-1}$, for $x \in X_n$, satisfying $\phi_{n,x}^0 = h_{n,x}(\phi_{n,\mathrm{rep}_n(x)}^0)$.*

Proof. This follows from Lemma 3.6.33 like Proposition 3.6.28. □

Proposition 3.6.35. *There is a family of colorings, $\phi_{n,x}^i \colon C_{n,n-1}(x) \to \mathfrak{I}_{n-1}$, for $x \in X_n$ and $i \in H_{n,x}$, satisfying the following properties:*

(i) *The coloring $\phi_{n,x}^0$ is strongly adapted.*

(ii) *We have $\phi_{n,x}^i = h_{n,x}(\phi_{n,\mathrm{rep}_n(x)}^i)$.*

(iii) *Each coloring $\phi_{n,x}^i$ is adapted.*

(iv) *There are sets $\mathcal{N}_{n,x}^i \subset C_{n-1}(x,10,r_n^\pm - 1)$ for $x \in X_n$ and $i \in H_{n,x}$, satisfying:*

   (a) $\mathcal{N}_{n,x}^i = \hat{h}_{n,x}(\mathcal{N}_{n,\widehat{\mathrm{rep}}_n(x)}^i)$,

   (b) $(\phi_{n-1,x}^i)^{-1}(4) = \mathcal{N}_{n,x}^i$, *and*

   (c) $\mathcal{N}_{n,x}^i \neq \mathcal{N}_{n,x}^j$ *if $i \neq j$.*

Proof. First, for $i = 0$, we take the strongly adapted colorings $\phi^0_{0,x}$ constructed in Proposition 3.6.28. So (i) is satisfied.

For every $x \in X_n^{\pm,\widehat{\Phi}}$, let $N_{n,x}$ be a maximal subset of $C_{n-1}(x, 10, r_n^{\pm}) \setminus \mathfrak{X}_{n-1}$ that is $r_{n-1}^2 s_{n-1}$-separated with respect to d_{n-2}. Choose an enumeration of the powerset $\mathcal{P}(N_{n,x})$,

$$\mathcal{P}(N_{n,x}) = \{\emptyset = \mathcal{N}^0_{n,x}, \mathcal{N}^1_{n,x}, \dots\}.$$

We have $|B_{n-1}(x, 10)| \leq (\deg X_{n-1})^{11}$ and $|C_{n,n-1}(x) \cap \mathfrak{X}_{n-1}| \leq 1$ by Corollary 2.2.12, Proposition 3.4.1 (vii), (2.5) and (3.79). Therefore

$$|C_{n-1}(x, 10, r_n^{\pm})| \geq |B_{n-1}(x, r_n^{\pm})| - (\deg X_{n-1})^{11} - 1.$$

By Lemma 2.1.5, $N_{n,x}$ is a $(r_{n-1}^2 s_{n-1} - 1)$-net in $|C_{n-1}(x, 10, r_0^{\pm})|$ with respect to d_{n-2}, so

$$|N_{n,x}| \geq \left\lfloor \frac{|B_{n-1}(x, r_n^{\pm})| - (\deg X_{n-1})^{11} - 1}{(\deg X_{n-2})^{r_{n-1}^2 s_{n-1}}} \right\rfloor \tag{3.80}$$

by Lemma 2.2.13, and therefore

$$|\mathcal{P}(N_{n,x})| \geq \exp_2 \left\lfloor \frac{|B_{n-1}(x, r_n^{\pm})| - (\deg X_{n-1})^{11} - 1}{(\deg X_{n-2})^{r_{n-1}^2 s_{n-1}}} \right\rfloor = \eta_n(|B_{n-1}(x, r_n^{\pm})|).$$

Thus an injective map $H_{n,x} \to \mathcal{P}(N_{n,x})$ is well defined by $i \mapsto \mathcal{N}^i_{n,x}$.

If $x \notin X_0^{\widehat{\Phi}}$, let $N_{n,x} = \hat{h}_{n,x}(N_{n,\mathrm{rep}_n(x)})$ and $\mathcal{N}^i_{n,x} = \hat{h}_{n,x}(\mathcal{N}^i_{n,\mathrm{rep}_n(x)})$, so that $N_{n,x}$ also satisfies (3.80). Then define

$$\phi^i_{n,x}(u) = \begin{cases} \phi^0_{n,x}(u) & \text{if } u \notin \mathcal{N}^i_{n,x} \\ 4 & \text{if } u \in \mathcal{N}^i_{n,x}. \end{cases}$$

With this definition, Property (i) is obvious because $\mathcal{N}^0_{n,x} = \emptyset$. Property (ii) follows immediately from Proposition 3.6.34 and the fact that, for $x \notin X_0^{\Phi}$, we have $\mathcal{N}^i_{0,x} = h_{0,x}(\mathcal{N}^i_{0,\mathrm{rep}_0(x)})$. Finally, (iv) follows since $\mathcal{N}^i_{n,x} \neq \mathcal{N}^j_{n,x}$ for $i \neq j$. □

In Section 3.6.1, it was said that $I_{n,x} \times I_{n,x}$ is considered as an initial segment of $H_{n,x}$ for every $x \in X_n$. Let $\iota_{n,x}$ denote the inclusion $I_{n,x} \times I_{n,x} \hookrightarrow H_{n,x}$. From now on, the notation $\phi^{i,j}_{n,x}$ will refer to the coloring $\phi^{\iota_{n,x}(i,j)}_{n,x}$.

3.6.7 Colorings ψ^N_n

Definition 3.6.36. Let $n \in \mathbb{N}$ and $x \in X_n$. A coloring $\psi \colon C_{n,-1}(x) \to [\Delta]$ is *rigid* if, for all $u \in C_{n,0}(x)$, there is some $i \in H_{n,x}$ such that the restriction of ψ to $C_{0,-1}(u)$ equals $\phi^i_{0,x}$.

Lemma 3.6.37. *For all $x_1, x_2 \in X_n^+$, if $d_n(x_1, x_2) \leq 2$, then $d_{n-1}(x_1, x_2) < r_n^+ s_n$.*

Proof. By the definition of E_n, there is a point $x_3 \in X_n$ and points, $u_1 \in \overline{C}_{n,n-1}(x_1)$, $u_2 \in \overline{C}_{n,n-1}(x_2)$ and $u_3, u_3' \in \overline{C}_{n,n-1}(x_3)$, such that $u_1 E_{n-1} u_3$ and $u_3' E_{n-1} u_2$. By Lemma 3.4.16, the triangle inequality, (3.16) and (3.45), we get

$$d_{n-1}(x_1, x_2) \leq 4R_n^+ + 2 = 4(r_n(2s_n + 3)) + 2 \leq 20 r_n s_n < r_n s_n^2,$$

since $s_n > 20$ by (3.3) and (3.11). □

Lemma 3.6.38. *For all $x_1, x_2, x_3 \in X_n^-$, if $x_1 E_n x_2 E_n x_3$, then $d_{n-1}(x_1, x_3) < r_n^- s_n$.*

Proof. By the definition of E_n, there are points, $u_1 \in \overline{C}_{n,n-1}(x_1)$, $u_2, u_2' \in \overline{C}_{n,n-1}(x_2)$ and $u_3 \in \overline{C}_{n,n-1}(x_3)$, such that $u_1 E_{n-1} u_2$ and $u_2' E_{n-1} u_3$. By Lemma 3.4.16, the triangle inequality, (3.16) and (3.45), we get

$$d_{n-1}(x_1, x_2) \leq 4R_n^- + 2 = 4(4r_n + 2) + 2 \leq 26 r_n < r_n s_n,$$

since $s_n > 26$ by (3.3) and (3.11). □

Proposition 3.6.39. *For $n \in \mathbb{N}$ and $x \in X_n^\pm$, let $A = C_{n,-1}(x)$ (respectively, $A = B_{n-1}(x, r_n^\pm - 1)$), let $\zeta \colon \bigcup_{a \in A} C_{n-1,-1}(a) \to [\Delta]$ be rigid coloring, and let $f \colon x \to x$ be an n-equivalence (respectively, an n-weak equivalence) preserving ζ. Then f is the identity map on A.*

Proof. We proceed by induction on $n \in \mathbb{N}$. If $n = 0$, then the result follows from Proposition 3.6.29 (iv). Therefore suppose that $n > 0$ and the result is true for $0 \leq m < n$. By hypothesis, f is an n-(weak) equivalence and $f(x) = x$. Thus, $f(C_{n-1,n-2}(x)) = C_{n-1,n-2}(x)$ and $f : x \to x$ is an $(n-1)$-equivalence by Definitions 3.6.6 (v) and 3.6.16 (iv). Hence f is the identity on $C_{n-1,n-2}(x)$ by the induction hypothesis.

Let us prove that f is the identity on $C_{n-1,n-2}(u)$ by induction on $u \in A$, using $\trianglelefteq_{n,x}$. The case $u = x$ follows from the induction hypothesis, so suppose $u \neq x$. By the induction hypothesis, we have $f(\mathrm{Pa}_{n,x}(u)) = \mathrm{Pa}_{n,x}(u)$. If $u \in X_n^\pm$, then $f(u) E_{n-1} f(\mathrm{Pa}_{n,x}(u)) = \mathrm{Pa}_{n,x}(u)$ by Definition 3.6.6 (iii). We consider the following cases.

If $u, \mathrm{Pa}_{n,x}(u) \in X_{n-1}^+$, then $d_{n-1}(u, f(u)) < r_n^+ s_n$ by Lemma 3.6.37. If $\mathrm{Pa}_{n,x}(u) \in X_{n-1}^-$, then $d_{n-1}(u, f(u)) < r_n^+ s_n$ by Lemma 3.6.38. By Definition 3.6.6 (iii), we have $\chi_{n-1}(u) = \chi_{n-1}(f(u))$. Thus Proposition 3.6.1 (ii) yields $f(u) = u$ in these two cases.

Finally, suppose that $u, f(u) \in X_n^-$ and $\mathrm{Pa}_{n,x}(u) \in X_n^+$. By the definition of E_{n-1}, there is some $u' \in X_{n-1}^+ \cap B_{n-1}(\mathrm{Pa}_{n,x}(u), 1)$ such that there are $v \in \overline{C}_{n-1,n-2}(u), v' \in C_{n-1,n-2}(u')$ with $v E_{n-2} v'$. Using the same argument as before, we get that f is the identity on $C_{n-1,n-2}(u')$; in particular $f(v') = v'$. Therefore $d_{n-2}(v, f(v)) \leq 2$, and we obtain $d_{n-2}(u, f(u)) \leq 2R_n^- + 2$. Then $f(u) = u$ as before, and we get that f is the identity on $C_{n-1,-1}(u)$ by the induction hypothesis. □

Corollary 3.6.40. *For $n \in \mathbb{N}$ and $x, y \in X_n^\pm$, let $A = C_{n,-1}(x)$ (respectively, $A = B_{n-1}(x, r_n^\pm - 1)$), let $\zeta\colon \bigcup_{a\in A} C_{n-1,-1}(a) \to [\Delta]$ and $\zeta\colon \bigcup_{b\in f(A)} C_{n-1,-1}(b) \to [\Delta]$ be rigid colorings, and let $f\colon x \to y$ and let $f\colon x \to y$ be a color-preserving n-equivalence (respectively, a color-preserving n-weak equivalence). Then $f = h_{n,y} \circ h_{n,x}^{-1}$ (respectively, $f = \hat{h}_{n,y} \circ \hat{h}_{n,x}^{-1}$).*

Definition 3.6.41. For $N \in \mathbb{N}$, let $\psi_n^N\colon X_n \to \mathfrak{I}_n^2$ and $\psi_{-1}^N\colon X_{-1} \to [\Delta]$ be defined by reverse induction on $n = -1, \dots, N$ as follows:

- For $n = N$, let $\psi_N^N = (\chi_N, 0)$.
- For $0 \le n < N$, define ψ_n^N so that, for every $x \in X_{n+1}$,
$$\psi_n^N|_{C_{n+1,n}(x)} = \left(\phi_{n,x}^{\psi_{n+1}^N(x)}, \chi_n(x)\right).$$
- Finally, define ψ_{-1}^N so that, for every $x \in X_0$,
$$\psi_{-1}^N|_{C_{0,-1}(x)} = \phi_{-1,x}^{\psi_0^N(x)}.$$

Remark 3.6.7. It follows from Proposition 3.6.1 (ii) that $\psi_n^N(x) \ne \psi_n^N(y)$ for $x, y \in X_n^\pm$ if $0 < d_{n-1}(x, y) < r_n^\pm s_n$.

Remark 3.6.8. By Definitions 3.6.1 (i) and 3.6.31 (i), for all $0 \le m \le N$ and $x \in X_m$, the value $\psi_m^N(x)$ determines whether x is in $\mathfrak{X}_m$.

Let $W_0 = 10$ and $W_i = 2$ for $i > 0$, and let Υ_n be recursively defined by

$$\Upsilon_{-1} = 0 \qquad \Upsilon_n = \Upsilon_{n-1} + \Lambda_{n-1}(W_n + 3R_n^+ + 1) + \Gamma_n^+ + \Lambda_n. \tag{3.81}$$

Lemma 3.6.42. *Fix $0 \le n \le N$ and $R > \Upsilon_n$. Let $A \subset X$ and $x \in A$ be such that $B_{-1}(x, R) \subset A$, and let $f\colon (A, x, \psi_{-1}^N) \to (f(A), f(x), \psi_{-1}^N)$ be a pointed colored graph isomorphism with respect to the restriction of E_{-1}. Then the following properties hold for $0 \le m \le n$ and $0 \le l \le n+1$:*

(i) *The restriction*
$$f\colon \left(X_{l-1} \cap B_{-1}(x, R - \Upsilon_{l-1}), x, \psi_{l-1}^N\right) \to \left(X_{l-1} \cap B_{-1}(f(x), R - \Upsilon_{l-1}), f(x), \psi_{l-1}^N\right)$$
is a pointed colored graph isomorphism with respect to E_{l-1}.

(ii) *For any $z \in X_{m-1} \cap B_{-1}(x, R - \Upsilon_{m-1} - \Lambda_{m-1} W_m)$, we have $z \in X_m^\pm$ if and only if $f(z) \in X_m^\pm$.*

(iii) *For all $z \in X_m \cap B_{-1}(x, R - \Upsilon_{m-1} - \Lambda_{m-1}(W_m + r_m^+))$, the restriction of f is an m-weak equivalence.*

(iv) *For any* $z \in X_m \cap B_{-1}(x, R - \Upsilon_{m-1} - \Lambda_{m-1}(W_m + r_m^+))$, *we have* $\psi_m^N(z) = \psi_m^N(f(z))$.

(v) *For any* $z \in X_m \cap B_{-1}(x, R - \Upsilon_{m-1} - \Lambda_{m-1}(W_m + r_m^+ + 1))$, *we have* $z \in \mathfrak{X}_m$ *if and only if* $f(z) \in \mathfrak{X}_m$.

(vi) *For all* $z \in X_{m-1} \cap B_{-1}(x, R - \Upsilon_{m-1} - \Lambda_{m-1}(W_m + 2R_m^+))$, *we have* $z \in \mathfrak{Z}_{m-1}^{\pm}$ *if and only if* $f(z) \in \mathfrak{Z}_{m-1}^{\pm}$.

(vii) *For any* $z \in X_m \cap B_{-1}(x, R - \Upsilon_{m-1} - \Lambda_{m-1}(W_m + 3R_m^+))$, *we have* $f(\overline{C}_{m,m-1}(z)) = \overline{C}_{m,m-1}(f(z))$.

(viii) *For all* $z \in X_m \cap B_{-1}(x, R - \Upsilon_{m-1} - \Lambda_{m-1}(W_m + 3R_m^+) - \Lambda_m)$, *we have* $f(C_{m,m-1}(z)) = C_{m,m-1}(f(z))$.

(ix) *For all* $z, z' \in X_m \cap B_{-1}(x, R - \Upsilon_{m-1} - \Lambda_{m-1}(W_m + 3R_m^+ + 1) - \Gamma_m^+)$, *we have* zE_mz' *if and only if* $f(z)E_mf(z')$.

(x) *For all* $z \in X_m \cap B_{-1}(x, R - \Upsilon_{m-1} - \Lambda_{m-1}(W_m + 3R_m^+ + 1) - \Gamma_m^+ - \Lambda_m)$, *the restriction of* f *to* $\bigcup_{u \in B_m(z,1)} \overline{C}_{m,-1}(u)$ *is an* m*-equivalence.*

Proof. We proceed by induction on m and l. For $l = 0$, property (i) is true by hypothesis. When $l > 0$, (i) follows from (3.81) and the induction hypothesis for $m = l - 1$ with (iv) and (ix). Thus, for $m = 0, \dots, n$, we will derive properties (ii)–(ix) from (i), completing the proof of the lemma.

Let us prove (ii). The coloring ψ_{m-1} is adapted by Remark 3.6.7. For every $z \in X_{m-1}$, we have $z \in X_m^{\pm}$ if and only if the colored set $(B_{m-1}(z, W_m/2), \phi_{m-1})$ has one of the patterns described in Definition 3.6.26 (i) and Lemma 3.6.32. By Proposition 3.4.1 (vi) and the triangle inequality, we get

$$B_{m-1}(z, W_m) \subset B_{-1}(z, \Lambda_{m-1}W_m) \subset B_{-1}(x, R - \Upsilon_m).$$

Therefore, the restriction $f\colon B_{m-1}(z, W_m/2) \to B_{m-1}(f(z), W_m/2)$ is an isometry by Lemma 2.2.3. The induction hypothesis with (i) implies that the set $B_{m-1}(z, W_m/2)$ has one of the patterns described in Definition 3.6.26 (i) and Lemma 3.6.32 if and only if $B_{m-1}(f(z), W_m/2)$ does. Then (ii) follows from (i).

To prove (iii), let $z \in X_m^{\pm}$. If $m = 0$, the result is obvious, so suppose $m > 0$. We have $f(z) \in X_m^{\pm}$ by (ii). By Proposition 3.4.1 (vi), we have

$$B_{m-1}(z, r_m^{\pm}) \subset B_{-1}(z, \Lambda_{m-1}r_m^+) \subset B_{-1}(x, R - \Upsilon_{m-1} - \Lambda_{m-1}W_m).$$

The properties Definition 3.6.16 (i),(ii) follow from (i). Definition 3.6.16 (iii) holds by the induction hypothesis with (v), and Definition 3.6.16 (iv) follows from the induction hypothesis with (x).

Let us prove (iv). By Definition 3.6.41, the restriction of ψ^N_{m-1} to $C_{m,m-1}(z)$ equals $\phi^i_{m-1,x}$ for some $i \in H_{m,x}$. Then $\psi_m(x) = \psi_m(y)$ if and only if the restrictions of ψ^N_{m-1} to $C_{m,m-1}(z)$ and $C_{m,m-1}(f(z))$ equal $\phi^i_{m-1,z}$ and $\phi^i_{m-1,f(z)}$. Moreover i is determined by the set $(\psi^N_{m-1})^{-1}(4) \cap B_{m-1}(z, r^\pm_m - 1) = \mathcal{N}^i_{m,x}$ if $m > 0$, or $(\psi^N_{-1})^{-1}(0) \cap C_{-1}(z, 10, r^\pm_0 - 1) = \mathcal{N}^i_{0,x}$ if $m = 0$. By (i), we have

$$
\begin{aligned}
f((\psi^N_{m-1})^{-1}(4) \cap B_{m-1}(z, r^\pm_m - 1)) &= (\psi^N_{m-1})^{-1}(4) \cap B_{m-1}(f(z), r^\pm_m - 1) && \text{if } m > 0, \\
f((\psi^N_{-1})^{-1}(0) \cap C_{-1}(z, 10, r^\pm_0 - 1)) &= (\psi^N_{-1})^{-1}(0) \cap C_{-1}(f(z), 10, r^\pm_0 - 1) && \text{if } m = 0.
\end{aligned}
$$

Then (iv) follows from Proposition 3.6.35 (a).

Property (v) follows from (iv) and Remark 3.6.8.

Let us prove (vi). Let $z \in B_{m-1}(x, R - \Upsilon_{m-1} - \Lambda_{m-1}(W_m + 2R^+_m))$. By (i), Proposition 3.4.1 (vi) and Lemma 2.2.15, we have that the restriction of f to

$$
B_{m-1}(x, R - \Upsilon_{m-1} - \Lambda_{m-1}(W_m + R^+_m))
$$

preserves $X^\pm_n$ and is an R^+_m-short scale isometry with respect to E_{m-1}. Then z satisfies (3.59) if and only if $f(z)$ does, and (vi) follows.

To prove (vii), let $z \in X_m \cap B_{m-1}(x, R - \Upsilon_{m-1} - \Lambda_{m-1}(W_m + 3R^+_m))$. By Lemma 3.4.16, we have $\overline{C}_{m,m-1}(z) \subset B_{m-1}(z, R^+_n)$. Using Proposition 3.4.1 (vi) and the triangle inequality, we get

$$
\overline{C}_{m,m-1}(z) \subset B_{m-1}(x, R - \Upsilon_{m-1} - \Lambda_{m-1}(W_m + 2R^+_m)).
$$

Therefore, for every $u \in \overline{C}_{m,m-1}(z)$, we have $u \in \mathcal{Z}^\pm_{m-1}$ if and only if $f(u) \in \mathcal{Z}^\pm_{m-1}$ by (vi). Let $y \in X_m$ satisfy $d_{m-1}(u, X_m) = d_{m-1}(u, y)$. Then $d_{m-1}(u, y) \leq R^+_m$ by Proposition 3.4.1 (iv), yielding $d_{-1}(u, y) \leq \Lambda_{m-1} R^+_m$ by Proposition 3.4.1 (vi). By (i), (ii) and Lemma 2.2.15, we have $f(y) \in X^\pm_m$ if and only if $y \in X^\pm_m$ and $d_{m-1}(u, y) = d_{m-1}(f(u), f(y))$. Then (vii) follows by (3.60).

Let us prove (viii). By Proposition 3.4.1 (vi) and the triangle inequality, we get

$$
B_m(z, 1) \subset B_{-1}(z, \Lambda_m) \subset B_{-1}(x, R - \Upsilon_{m-1} - \Lambda_{m-1}(W_m + 3R^+_m))).
$$

Therefore, we have $f(\overline{C}_{n,n-1}(u)) = \overline{C}_{n,n-1}(f(u))$ for all $u \in B_m(z, 1)$ by (vii). Also, we have $\psi_m(u) = \psi_m(f(u))$ for all $u \in B_m(z, 1)$ by (iv). In particular, this yields $\chi_m(u) = \chi_m(f(u))$. Then the result follows from Proposition 3.6.1 (ii) and (3.65).

Property (ix) follows easily from (vii), Lemma 2.2.15 and the definition of E_m.

Finally, (x) follows from (i), (ii), (iv), (v) and the induction hypothesis with (x). □

Recall that ε_n is the sequence of positive integers defined in Section 3.2. Let $\delta_n = 2(4\Gamma^+_n + \Upsilon_n + 2\Lambda_n)$.

Proposition 3.6.43. *For $0 \leq n \leq N$ and $u \in X$, let*

$$f\colon \left(B_{-1}\left(u, \delta_n/2\right), u, \psi_{-1}^N\right) \to \left(B_{-1}\left(f(u), \delta_n/2\right), f(u), \psi_{-1}^N\right)$$

be a color-preserving pointed graph isomorphism with respect to E_{-1}. Then, we have that either $f(u) = u$, or $d_{-1}(u, f(u)) > \varepsilon_n$.

Proof. Let $x \in X_n^\pm$ such that $u \in C_{n,-1}(x)$. We have $d_{-1}(u, x) \leq \Gamma_n^+$ by Lemma 3.5.1, and, using the triangle inequality, we get $B_{-1}(x, 3\Gamma_n^+ + \Upsilon_n + 2\Lambda_n) \subset \operatorname{dom} f$. Then we have $f(x) \in X_n^\pm$ and $\psi_n^N(x) = \psi_n^N(f(x))$ by Lemma 3.6.42 (ii),(iv). In particular, $\chi_n(x) = \chi_n(f(x))$. Therefore, either $f(x) = x$, or $d_{n-1}(x, f(x)) \geq r_n^\pm s_n$ by Proposition 3.6.1 (ii).

If $f(x) = x$, then $f(u) = u$ by Proposition 3.6.39 and the result follows. So suppose $d_{n-1}(x, f(x)) \geq 2r_n^\pm s_n$. By Lemma 3.5.1, we have $d_{-1}(u, x) = d_{-1}(f(u), f(x)) \leq \Gamma_n^\pm$. Then, by the triangle inequality, we obtain $d(u, f(u)) \geq r_n^\pm s_n - 2\Gamma_n^\pm$. Applying now Lemma 3.2.1, we get $d(u, f(u)) \geq \varepsilon_n$. □

Then Proposition 3.6.43 gives the following result.

Theorem 3.6.44. *Let X be a connected infinite simple graph with bounded degree $\Delta = \deg X < \infty$, and let ε_n be an increasing sequence of positive integers. Then there are constants δ_n, with δ_n large enough depending only on Δ, ε_m for $m \leq n$, and δ_m for $m < n$, such that for every $N \in \mathbb{N}$ there is a coloring ψ^N of X by Δ colors, satisfying*

$$\forall x, y \in X,\ \forall n \leq N, \quad 0 < d(x, y) < \varepsilon_n \Rightarrow [B(x, \delta_i), x, \psi^N] \neq [B(y, \delta_i), y, \psi^N]. \quad (3.82)$$

Proposition 3.6.45. *For $n = 0, \ldots, N$, $x \in \mathfrak{X}_n$ and $u \in C_{n,m}(p)$, we have $\psi_m^N(u) = \psi_m^N(\mathfrak{h}_{n,x}(u))$ for $-1 \leq m \leq n$.*

Proof. We proceed by inverse induction on m. For $m = N$, we have $\psi_N^N = (\chi_N, 0)$. So $\psi_N^N(u) = \psi_N^N(\mathfrak{h}_{n,x}(u))$ by Proposition 3.6.1 (iii).

Suppose that, for $0 \leq m < N - 1$, the result is true for $m + 1$. Let $u \in C_{n,m}(p)$, $z \in C_{n,m+1}(p)$ such that $u \in C_{m+1,m}(z)$. By the induction hypothesis, $\psi_{m+1}^N(z) = \psi_{m+1}^N(\mathfrak{h}_{n,x}(z))$. By the definition of ψ_{m+1}^N, Lemmas 3.6.4 and 3.6.9 and Corollary 3.6.40, this means that the restrictions of ψ_{m+1} to $C_{m+1,m}(z)$ and $C_{m+1,m}(\mathfrak{h}_{n,x}(z))$ equal $\phi_{m,x}^{i,j}$ and $\phi_{m,\mathfrak{h}_{n,x}(z)}^{i,j}$ for some $i, j \in H_{m,x}$. But $\phi_{m,\mathfrak{h}_{n,x}(z)}^{i,j} = \mathfrak{h}_{n,x}(\phi_{m,x}^{i,j})$ by Proposition 3.6.35 (ii). □

If we now set $\alpha_n = 2\mathfrak{s}_n + \mathfrak{t}_n + 3\omega_n$, then Proposition 3.6.45 together with Corollaries 3.3.8 and 3.5.3 give the following result.

Theorem 3.6.46. *Let X and ε_n be like in Theorem 3.1.1, and let $p \in X$ be a distinguished point. Suppose that, for large enough constants $\mathfrak{r}_n$, recursively depending on Δ, on ε_m for $m \leq n$, and on $\mathfrak{r}_m$ and ω_m for $m < n$, the sets*

$$\Omega_n := \{\, x \in X \mid [B(x, \mathfrak{r}_n), x, d_X] = [B(p, \mathfrak{r}_n), p, d_X] \,\}$$

are ω_n-nets in X for some constants ω_n. Then, for some large enough positive integers r_n, depending on Δ, on ε_m for $m \leq n$, and on r_m for $m < n$, and for every $N \in \mathbb{N}$, there is a coloring ψ^N by Δ colors satisfying (3.82) *and such that the sets*

$$\widehat{\Omega}_n := \left\{\, x \in X \mid \left[B\left(x, \textstyle\sum_{j=0}^n r_j\right), x, \phi\right] = \left[B\left(p, \textstyle\sum_{j=0}^n r_j\right), p, \phi\right] \,\right\}, \quad \forall n \leq N,$$

are α_n-nets in X for some positive integers α_n.

3.6.8 The coloring ϕ

We will derive Theorems 3.1.1 and 3.1.2 from Theorems 3.6.44 and 3.6.46. Let X be a graph and ε_n be an increasing sequence of positive integers satisfying the conditions of Theorem 3.6.44 (respectively, Theorem 3.6.46). Then this result gives a sequence of colorings ψ^N. The set of colorings of X by $\deg X$ colors is endowed with the topology of convergence over finite subsets of X. Since the set $[\deg X]$ of colors is finite, possibly passing to a subsequence, we can suppose that the sequence of colorings ψ^N given by Theorems 3.6.44 and 3.6.46 converge to some coloring ϕ. Then, for any finite $A \subset X$, the colorings ϕ and ψ^m coincide on A for m large enough. We will prove that ϕ satisfies Theorem 3.1.1 (respectively, Theorem 3.1.2).

Assume by absurdity that there are $n \in N$, $x, y \in X$ such that $0 < d(x, y) < \varepsilon_n$ and $[B(x, \delta_i), x, \phi] \neq [B(y, \delta_i), y, \phi]$. By the convergence of the ψ^N, there is some $m > n$ such that $[B(x, \delta_n), x, \phi] = [B(x, \delta_n), x, \psi^m]$ and $[B(y, \delta_n), y, \phi] = [B(y, \delta_n), y, \psi^m]$, contradicting (3.82). Therefore ϕ satisfies Theorem 3.1.1, with the same choice of sequence δ_n.

Suppose that additionally, the family ψ^N satisfies the conditions of Theorem 3.6.46, with p the distinguished point. Then, for any $n \leq N$ and $x \in X$, there is some $y \in X$ such that $d(x, y) \leq \alpha_n$ and $[B(y, \sum_{i=0}^n r_i), y, \psi^N] = [B(p, \sum_{i=0}^n r_i), p, \psi^N]$. Assume by absurdity that there are $n \in \mathbb{N}$ and $x \in X$ such that, for all $y \in B(x, \alpha_n)$, we have $[B(y, \sum_{i=0}^n r_i), y, \phi] \neq [B(p, \sum_{i=0}^n r_i), p, \phi]$. By the convergence of ψ^N, we have that ϕ and ψ^m coincide over $B(p, \sum_{i=0}^n r_i)$ and $B(y, \sum_{i=0}^n r_i)$ for every $y \in B(x, \alpha_n)$ for m large enough, a contradiction. Therefore, the sets

$$\left\{\, x \in X \mid \left[B\left(x, \textstyle\sum_{i=0}^n r_j\right), x, \phi\right] = \left[B\left(p, \textstyle\sum_{i=0}^n r_j\right), p, \phi\right] \,\right\}$$

are α_n-nets in X. Therefore ϕ satisfies Theorem 3.1.1, with the same choice of sequence α_n.

The common idea is that the conditions for limit-aperiodicity and repetitivity can be checked locally around every point, and since the family ψ^N satisfies this conditions with uniform bounds on the involved constants, they pass to the limit.

Part II

Realization of Riemannian manifolds as leaves

Chapter 4

Preliminaries on foliated spaces and Riemannian geometry

4.1 Foliated spaces

Standard references for foliated spaces are [57], [18, Chapter 11], [19, Part 1] and [34].

Let Z be a space and let U be an open set in $\mathbb{R}^n \times Z$ ($n \in \mathbb{N}$), with coordinates (x, z). For $m \in \mathbb{N}$, a map $f : U \to \mathbb{R}^p$ ($p \in \mathbb{N}$) is of *class* C^m if its partial derivatives up to order m with respect to x exist and are continuous on U. If f is of class C^m for all m, then it is called of *class* C^∞. Let Z' be another space, and let $h : U \to \mathbb{R}^p \times Z'$ ($p \in \mathbb{N}$) be a map of the form $h(x, z) = (h_1(x, z), h_2(z))$, for maps $h_1 : U \to \mathbb{R}^p$ and $h_2 : \mathrm{pr}_2(U) \to Z'$, where $\mathrm{pr}_2 : \mathbb{R}^n \times Z \to Z$ is the second factor projection. It will be said that h is of *class* C^m if h_1 is of class C^m and h_2 is continuous.

For $m \in \mathbb{N} \cup \{\infty\}$ and $n \in \mathbb{N}$, a *foliated structure* $\mathcal{F}$ of *class* C^m and *dimension* $\dim \mathcal{F} = n$ on a space X is defined by a collection $\mathcal{U} = \{(U_i, \phi_i)\}$, where $\{U_i\}$ is an open covering of X, and each ϕ_i is a homeomorphism $U_i \to B_i \times Z_i$, for a locally compact Polish space Z_i and an open ball B_i in $\mathbb{R}^n$, such that the coordinate changes $\phi_j \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ are locally C^m maps of the form

$$\phi_j \phi_i^{-1}(x, z) = (g_{ij}(x, z), h_{ij}(z)) \ .$$

These maps h_{ij} will be called the *local transverse components* of the changes of coordinates. Each (U_i, ϕ_i) is called a *foliated chart*, the sets $\phi_i^{-1}(B_i \times \{z\})$ ($z \in Z_i$) are called *plaques*, and the collection $\mathcal{U}$ is called a *foliated atlas* of *class* C^m. Two C^m foliated atlases on X define the same C^m foliated structure if their union is a C^m foliated atlas. If we consider foliated atlases so that the sets Z_i are open in some fixed space, then $\mathcal{F}$ can be also described as a maximal foliated atlas of class C^m. The term *foliated space* (of *class* C^m) is used for $X \equiv (X, \mathcal{F})$. If no reference to the class C^m is indicated, then it is understood that X is a C^0 (or *topological*) foliated space. The concept of C^m foliated space can be extended to the case *with boundary* in the obvious way, and the boundary of a C^m foliated space is a C^m foliated space without boundary.

The foliated structure of a space X induces a locally Euclidean topology on X, the basic open sets being the plaques of all foliated charts, which is finer than the original topology. The connected components of X in this topology are called *leaves*. Each leaf is a connected C^m n-manifold with the differential structure canonically induced by $\mathcal{F}$. The leaf that contains each point $x \in X$ is denoted by L_x. The leaves of $\mathcal{F}$ form a partition of X that determines the topological foliated structure. The corresponding quotient space, called *leaf space*, is denoted by $X/\mathcal{F}$.

The restriction of $\mathcal{F}$ to some open subset $U \subset X$ is the foliated structure $\mathcal{F}|_U$ on U defined by the charts of $\mathcal{F}$ whose domains are contained in U. More generally, a subspace $Y \subset X$ is a C^m *foliated subspace* when it is a subspace with a C^m foliated structure $\mathcal{G}$ so that, for each $y \in Y$, there is a foliated chart of $\mathcal{F}$ defined on a neighborhood U of y in X, whose restriction to $U \cap Y$ can be considered as a chart of $\mathcal{G}$ in the obvious way; in particular, $\dim \mathcal{G} \leq \dim \mathcal{F}$. For instance, any saturated subspace is a C^m foliated subspace.

A map between foliated spaces is said to be a *foliated map* if it maps leaves to leaves. A foliated map between C^m foliated spaces is said to be of *class* C^m if its local representations in terms of foliated charts are of class C^m. A C^m *foliated diffeomorphism* between C^m foliated spaces is a C^m foliated homeomorphism between them whose inverse is also a C^m foliated map.

Any topological space is a foliated space whose leaves are its points. On the other hand, any connected C^m n-manifold M is a C^m foliated space of dimension n with only one leaf. The C^m foliated maps $M \to X$ can be considered as C^m maps to the leaves of X, and may be also called C^m *leafwise maps*. They form a set denoted by $C^m(M, \mathcal{F})$, which can be equipped with the obvious generalization of the (*weak*) C^m *topology*. In particular, for $m = 0$, we get the subspace $C(M, \mathcal{F}) \subset C(M, X)$ with the compact-open topology. For instance, $C(I, \mathcal{F})$ ($I = [0, 1]$) is the space of leafwise paths in X.

Many concepts of manifold theory readily extend to foliated spaces. In particular, if $\mathcal{F}$ is of class C^m with $m \geq 1$, there is a vector bundle $T\mathcal{F}$ over X whose fiber at each point $x \in X$ is the tangent space T_xL_x. Observe that $T\mathcal{F}$ is a foliated space of class C^{m-1} with leaves TL for leaves L of X. Then we can consider a C^{m-1} Riemannian structure on $T\mathcal{F}$, which is called a (*leafwise*) *Riemannian metric* on X. This is a section of the associated bundle over X of positive definite symmetric bilinear forms on the fibers of $T\mathcal{F}$, which is C^{m-1} as foliated map. In this paper, a *Riemannian foliated space* is a C^∞ foliated space equipped with a C^∞ Riemannian metric, and an *isometry* between Riemannian foliated spaces is a C^∞ diffeomorphism between them whose restrictions to the leaves are isometries; in this case, the Riemannian foliated spaces are called *isomertric*.

A foliated space has a "transverse dynamics," which can be described by using a pseudogroup (see [40–42]). A *pseudogroup* $\mathcal{H}$ on a space is a maximal collection of homeomorphisms between open subsets of Z that contains id_Z, and is closed by the operations of composition, inversion, restriction to open subsets of their domains, and combination. This is a generalization of a dynamical system, and all basic dynamical concepts can be directly generalized to pseudogroups. For instance, we can consider its *orbits*, and the corresponding orbit space is denoted by $Z/\mathcal{H}$. It is said that $\mathcal{H}$ is *generated* by a subset E when all of its elements can be obtained from the elements of E by using the pseudogroup operations. Certain *equivalence* relation between pseudogroups was introduced [40], [41], and equivalent pseudogroups should be considered to represent the same dynamics; in particular, they have homeomorphic orbit spaces.

The *germ groupoid* of $\mathcal{H}$ is the topological groupoid of germs of maps in $\mathcal{H}$ at all points of their domains, with the operation induced by the composite of partial maps and the étale topology. Its subspace of units can be canonically identified with Z. For each $x \in Z$, the group of elements of this groupoid whose source and range is x is called the *germ group* of $\mathcal{H}$ at x. The germ groups at points in the same orbit are conjugated in the germ groupoid, and therefore the *germ group* of each orbit is defined up to isomorphisms. Under pseudogroup equivalences, corresponding orbits have isomorphic germ groups.

Let $\mathcal{U} = \{U_i, \phi_i\}$ be a foliated atlas of $\mathcal{F}$, with $\phi_i : U_i \to B_i \times Z_i$, and let $p_i = \mathrm{pr}_2 \phi_i : U_i \to Z_i$. The local transverse components of the corresponding changes of coordinates can be considered as homeomorphisms between open subsets of $Z = \bigsqcup_i Z_i$, which generate a pseudogroup $\mathcal{H}$. The equivalence class of $\mathcal{H}$ depends only on $\mathcal{F}$, and is called its *holonomy pseudogroup*. There is a canonical homeomorphism between the leaf space and the orbit space, $X/\mathcal{F} \to Z/\mathcal{H}$, given by $L \mapsto \mathcal{H}(p_i(x))$ if $x \in L \cap U_i$.

The *holonomy groups* of the leaves are the germ groups of the corresponding $\mathcal{H}$-orbits. The leaves with trivial holonomy groups are called *leaves without holonomy*. The union of leaves without holonomy is denoted by X_0. If X is second countable, then X_0 is a dense G_δ saturated subset of X [30, 43].

Given a loop α in a leaf L with base point x, there is a partition $0 = t_0 < t_1 < \cdots < t_k = 1$ of I and there are foliated charts $(U_{i_1}, \phi_{i_1}), \dots, (U_{i_k}, \phi_{i_k})$ such that $\alpha([t_{l-1}, t_l]) \subset U_{i_l}$ for $l \in \{1, \dots, k\}$. We can assume $(U_{i_k}, \phi_{i_k}) = (U_{i_1}, \phi_{i_1})$ because α is a loop. Let h_{i_{l-1},i_l} be the local transverse component of each change of coordinates $\phi_{i_l}\phi_{i_{l-1}}^{-1}$ defined around $p_{i_{l-1}}c(t_{l-1})$ and with $h_{i_{l-1},i_l}p_{i_{l-1}}\alpha(t_{l-1}) = p_{i_l}\alpha(t_l)$. The germ the composition $h_{i_{k-1},i_k} \cdots h_{i_1,i_0}$ at $p_{i_0}(x) = p_{i_k}(x)$ depends only on $\mathcal{F}$ and the class of α in $\pi_1(L, x)$, obtaining a surjective homomorphism of $\pi_1(L,x)$ to the holonomy group of L. This homomorphism defines a connected covering $\widetilde{L}^{\mathrm{hol}}$ of L, which is called its *holonomy covering*.

Now, let R be an equivalence relation on a topological space X. A subset of X is called (R-) *saturated* if it is a union of (R-) equivalence classes. The equivalence relation R is said to be (*topologically*)*transitive* if there is an equivalence class that is dense in X. A subset $Y \subset X$ is called an (R-) *minimal set* if it is a minimal element of the family of nonempty saturated closed subsets of X ordered by inclusion; this is equivalent to the condition that all equivalence classes in Y are dense in Y. In particular, X (or R) is called *minimal* when all equivalence classes are dense in X. These concepts apply to foliated spaces with the equivalence relation whose equivalence classes are the leaves.

4.2 Riemannian geometry

Let M be a Riemannian manifold possibly with boundary or corners (in the sense of [22], [27]). Connectedness of Riemannian manifolds is not assumed in Sections 4.2, 5.1 and 5.8 because it is not relevant for the concepts of these sections, but this property is assumed in the rest of the paper: it is needed in Section 5.2, and it is implicit in Sections 5.3–5.7 and 5.9–5.10 because the manifolds are given by elements of $\mathcal{M}_*(n)$. The following standard notation will be used. The metric tensor is denoted by g, the distance function on each of the connected components of M by d, the tangent bundle by $\pi : TM \to M$, the $\mathrm{GL}(n)$-principal bundle of tangent frames by $\pi : PM \to M$, the $\mathrm{O}(n)$-principal bundle of orthonormal tangent frames by $\pi : QM \to M$, the Levi-Civita connection by ∇, the curvature by $\mathcal{R}$, the exponential map by $\exp : TM \to M$ (if M is complete and $\partial M = \emptyset$), the open and closed balls of center $x \in M$ and radius $r > 0$ by $B(x,r)$ and $\overline{B}(x,r)$, respectively, and the injectivity radius by inj (if $\partial M = \emptyset$). The *penumbra* around a subset $S \subset M$ of radius $r > 0$ is the set $\mathrm{Pen}(S,r) = \{\, x \in M \mid d(x,S) < r \,\}$. If needed, "$M$" will be added to all of the above notation as a subindex or superindex. When a family of Riemannian manifolds M_i is considered, we may add the subindex or superindex "i" instead of "M_i" to the above notation. A covering of M is assumed to be equipped with the lift of g.

For $m \in \mathbb{Z}^+$, let $T^{(m)}M = T \cdots TM$ (m times). We also set $T^{(0)}M = M$. If $l < m$, $T^{(l)}M$ is sometimes identified with a regular submanifold of $T^{(m)}M$ via zero sections, and therefore, for each $x \in M$, the notation x may be also used for the zero elements of T_xM, T_xTM, etc. When the vector space structure of T_xM is emphasized, its zero element is denoted by 0_x, or simply by 0, and the image of the zero section of $\pi : TM \to M$ is denoted by $Z \subset TM$. Let $\pi\colon T^{(m)}M \to T^{(l)}M$ be the vector bundle projection given by composing the tangent bundle projections; in particular, we have $\pi : T^{(m)}M \to M$.

Given any C^m map between Riemannian manifolds, $\phi : M \to N$, the induced map $T^{(m)}M \to T^{(m)}N$ will be denoted by $\phi_*^{(m)}$ (or simply ϕ_* if $m = 1$, ϕ_{**} if $m = 2$, and so on).

Banach manifolds are also considered in some parts of the paper, using analogous notation.

The Levi-Civita connection determines a decomposition $T^{(2)}M = \mathcal{H} \oplus \mathcal{V}$, as direct sum of the horizontal and vertical subbundles. The *Sasaki metric* on TM is the unique Riemannian metric $g^{(1)}$ so that $\mathcal{H} \perp \mathcal{V}$ and the canonical identities $\mathcal{H}_\xi \equiv T_\xi M \equiv \mathcal{V}_\xi$ are isometries for every $\xi \in TM$.

Continuing by induction, for $m \geq 2$, the *Sasaki metric* on $T^{(m)}M$ is defined by $g^{(m)} = (g^{(m-1)})^{(1)}$. The notation $d^{(m)}$ is used for the corresponding distance function on the connected components, and the corresponding open and closed balls of center $\xi \in T^{(m)}M$ and radius $r > 0$ are denoted by $B^{(m)}(\xi, r)$ and $\overline{B}^{(m)}(\xi, r)$, respectively. We may add the subindex "M" to this notation if necessary, or the subindex "i" instead of "M_i" when a family of Riemannian manifolds M_i is considered. From now on, $T^{(m)}M$ is assumed to be equipped with $g^{(m)}$.

Remark 4.2.1. The following properties hold for $l < m$ and $\pi : T^{(m)}M \to T^{(l)}M$:

(i) $g^{(m)}|_{T^{(l)}M} = g^{(l)}$.

(ii) The submanifold $T^{(l)}M \subset T^{(m)}M$ is totally geodesic and orthogonal to the fibers of π. This follows easily by induction on m, where the case $m = 1$ is proved in [64, Corollary of Theorem 13].

(iii) The projection π is a Riemannian submersion with totally geodesic fibers. Again, this follows by induction on m, and the case $m = 1$ is proved in [64, Theorems 14 and 18].

(iv) For every $\xi \in T^{(m)}M$, its projection $\pi(\xi)$ is the only point $\zeta \in T^{(l)}M$ that satisfies $d^{(m)}(\xi, \zeta) = d^{(m)}(\xi, T^{(l)}M)$. To see this, it is enough to prove that $\pi(\xi)$ is the only critical point of the distance function $d^{(m)}(\cdot, \xi)$ on $T^{(l)}M$. These critical points are just the points $\zeta \in T^{(l)}M$ where the shortest $g^{(m)}$-geodesics γ from ζ to ξ are orthogonal to $T^{(l)}M$ at ζ. Hence γ is a geodesic in $\pi^{-1}(\zeta)$ by (iii), obtaining $\zeta = \pi(\xi)$.

(v) For all $\zeta, \zeta' \in T^{(l)}M$, the point ζ' is the only $\xi \in \pi^{-1}(\zeta')$ satisfying $d^{(m)}(\xi, \zeta) = d^{(m)}(\xi, \pi^{-1}(\zeta))$. This follows like (iv), using (ii) instead of (iii).

Let $(U; x^1, \dots, x^n)$ be a chart of M. The corresponding metric coefficients are denoted by g_{ij}, and the Christoffel symbols of the first and second kind are denoted by Γ_{ijk} and Γ_{ij}^k, respectively. Using the Einstein notation, recall that

$$\Gamma_{ij}^{\alpha} g_{\alpha k} = \Gamma_{ijk} = \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}) \,. \tag{4.1}$$

Identify the functions x^i, g_{ij}, Γ_{ijk} and Γ_{ij}^k with their lifts to TU. Thus we get a chart $(U^{(1)}; x_{(1)}^1, \dots, x_{(1)}^{2n})$ of TM with $U^{(1)} = TU$, $x_{(1)}^i = x^i$ and $x_{(1)}^{n+i} = v^i$ for $1 \leq i \leq n$, where the functions v^i give the coordinates of tangent vectors with respect to the local frame $(\partial_1, \dots, \partial_n)$ of TU induced by $(U; x^1, \dots, x^n)$. The coefficients of the Sasaki metric $g^{(1)}$ with respect to $(TU; x_{(1)}^1, \dots, x_{(1)}^{2n})$ are [64, Eq. (3.5)]:

$$\left.\begin{aligned} g_{ij}^{(1)} &= g_{ij} - g_{\alpha\gamma}\Gamma_{\mu\beta}^{\alpha}\Gamma_{\alpha\nu}^{\beta} v^\mu v^\nu \\ g_{n+i\,j}^{(1)} &= \Gamma_{j\mu i} v^\mu \\ g_{n+i\,n+j}^{(1)} &= g_{ij} \end{aligned}\right\} \tag{4.2}$$

for $1 \leq i,j \leq n$. Thus the metric coefficients $g_{\alpha\beta}^{(1)}$ are given by universal fractional expressions of the functions g_{ij}, $\partial_k g_{ij}$ and v^i $(1 \leq i,j,k \leq n)$.

Using induction again, for $m \geq 2$, let $(U^{(m)}; x_{(m)}^1, \dots, x_{(m)}^{2^m n})$ be the chart of $T^{(m)}M$ induced by the chart $(U^{(m-1)}; x_{(m-1)}^1, \dots, x_{(m-1)}^{2^{m-1}n})$ of $T^{(m-1)}M$, and let $g_{\alpha\beta}^{(m)}$ be the corresponding coefficients of $g^{(m)}$.

Lemma 4.2.1. (i) *The coefficients $g_{\alpha\beta}^{(m)}$ are given by universal fractional expressions of the coordinates $x_{(m)}^{n+1}, \dots, x_{(m)}^{2^m n}$ and the partial derivatives up to order m of the coefficients g_{ij}.*

(ii) *For each $\rho > 0$, the partial derivatives up to order m of the coefficients g_{ij} are given by universal linear expressions of the functions $(\sigma_{\rho,\mu}^{(m)})^* g_{\alpha\beta}^{(m)}$ for $n+1 \leq \mu \leq 2^m n$, where $\sigma_{\rho,\mu}^{(m)} : U \to U^{(m)}$ is the section of $\pi : U^{(m)} \to U$ determined by $(\sigma_{\rho,\mu}^{(m)})^* x_{(m)}^\nu = \rho\delta_{\mu\nu}$ for $n+1 \leq \nu \leq 2^m n$, using Kronecker's delta.*

Proof. We proceed by induction on m. For $m = 1$, (i) holds by (4.1) and (4.2), and (ii) holds by the second and third equalities of (4.2), since $\partial_i g_{jk} = \Gamma_{ijk} + \Gamma_{ikj}$ by (4.1). For arbitrary $m \geq 2$, assuming that (i) and (ii) hold for the case $m-1$, we get both properties for m by applying the above case to $(g^{(m-1)})^{(1)} = g^{(m)}$. □

Let $\Omega \subset M$ be a compact domain and $m \in \mathbb{N}$. Fix a finite collection of charts of M that covers Ω, $\mathfrak{U} = \{(U_a; x_a^1, \dots, x_a^n)\}$, and a family of compact subsets of M with the same index set as $\mathcal{U}$, $\mathcal{K} = \{K_a\}$, such that $\Omega \subset \bigcup_a K_a$, and $K_a \subset U_a$ for all a. The corresponding C^m norm of a C^m tensor T on Ω is defined by

$$\|T\|_{C^m,\Omega,\mathcal{U},\mathcal{K}} = \max_a \max_{x\in K_a\cap\Omega} \sum_{|I|\le m} \sum_{J,K} \left| \frac{\partial^{|I|} T^K_{a,J}}{\partial x^I_a}(x) \right| ,$$

using the standard multi-index notation, where $T^K_{a,J}$ are the coefficients of T on $U_a \cap \Omega$ with respect to the frame induced by $(U_a; x^1_a, \dots, x^n_a)$. With this norm, the C^m tensors on Ω of a fixed type form a Banach space. By taking the projective limit as $m \to \infty$, we get the Fréchet space of C^∞ tensors of that type equipped with the C^∞ *topology* (see e.g. [44]). Observe that $\mathcal{U}$ and $\mathcal{K}$ are also qualified to define the norm $\| \ \|_{C^m,\Omega',\mathcal{U},\mathcal{K}}$ for any compact subdomain $\Omega' \subset \Omega$. It is well known that $\| \ \|_{C^m,\Omega,\mathcal{U},\mathcal{K}}$ is equivalent to the norm $\| \ \|_{C^m,\Omega,g}$ defined by

$$\|T\|_{C^m,\Omega,g} = \max_{0\le l\le m} \max_{x\in\Omega} |\nabla^l T(x)| ;$$

i.e., there is some $C \ge 1$, depending only on M, m, Ω, $\mathcal{U}$, $\mathcal{K}$ and g, such that

$$\frac{1}{C} \| \ \|_{C^m,\Omega,\mathcal{U},\mathcal{K}} \le \| \ \|_{C^m,\Omega,g} \le C \| \ \|_{C^m,\Omega,\mathcal{U},g} . \tag{4.3}$$

In particular, for $m = 0$ and $f \in C^\infty(M)$,

$$\|f\|_\Omega := \|f\|_{C^0,\Omega,\mathcal{U},\mathcal{K}} = \|f\|_{C^0,\Omega,g} = \max_{x\in\Omega} |f(x)| , \tag{4.4}$$

which is independent of the choices $\mathcal{U}$, $\mathcal{K}$ and g.

The norms $\| \ \|_{C^m,\Omega,\mathcal{U},\mathcal{K}}$ and $\| \ \|_{C^m,\Omega,g}$ have straightforward extensions to tensors with values in a separable Hilbert space $\mathbb{E}$, and satisfy the obvious versions of (4.3) and (4.4), and $C^k(M, \mathbb{E})$ is assumed to be equipped with the C^k topology ($k \in \mathbb{N} \cup \{\infty\}$).

For $f \in C^\infty(M, \mathbb{E})$, recall that $\nabla f = df$ (its de Rham differential). For each m, the map

$$f^{(m)}_* \equiv \left(f^{(m),1}_*, \dots, f^{(m),2^m}_*\right) : T^{(m)}M \to T^{(m)}\mathbb{E} \equiv \mathbb{E}^{2^m}$$

is also C^∞ and with values in a separable Hilbert space. In the following lemma, we consider the local representations of f and every $f^{(m),\lambda}_*$ with respect to coordinate systems $(U, x^1, \dots, x^n)$ and $(U^{(m)}, x^1_{(m)}, \dots, x^{2^m n}_{(m)})$ of M and $T^{(m)}M$. Moreover each function on M or U is identified with its lift to $T^{(m)}M$ or $U^{(m)}$.

Lemma 4.2.2. *The following properties hold:*

(i) *The local representation of every function $f^{(m),\lambda}_*$ is a universal polynomial expression of $x^{n+1}_{(m)}, \dots, x^{2^m n}_{(m)}$ and the partial derivatives up to order m of the local representation of f.*

(ii) *For each $\rho > 0$, the partial derivatives up to order m of the local representation of f are given by universal linear expressions of the functions $(\sigma_{\rho,\mu}^{(m)})^* f_*^{(m),\lambda}$ for $n+1 \le \mu \le 2^m n$, where $\sigma_{\rho,\mu}^{(m)} : U \to U^{(m)}$ is the section of $\pi : U^{(m)} \to U$ determined by*[1] *$(\sigma_{\rho,\mu}^{(m)})^* x_{(m)}^{\nu} = \rho\delta_{\mu\nu}$ for $n+1 \le \nu \le 2^m n$.*

Proof. By using induction on m, the result clearly boils down to the case $m = 1$. But, in this case, the statement follows because $f_* \equiv (f, df) : TM \to T\mathbb{E} \equiv \mathbb{E}^2$. □

When $\partial M = \emptyset$, it is said that M is of *bounded geometry* if $\mathrm{inj}_M > 0$ and the function $|\nabla^m \mathcal{R}|$ is bounded for all $m \in \mathbb{N}$; in particular, M is complete since $\mathrm{inj}_M > 0$. More precisely, given $r > 0$ and a sequence $C_m > 0$, if $\mathrm{inj}_M \ge r$ and $|\nabla^m \mathcal{R}| \le C_m$ for all $m \in \mathbb{N}$, then (r, C_m) is called a *geometric bound* of M. A family $\mathcal{C}$ of Riemannian manifolds without boundary is called of *equi-bounded geometry* if all of them are of bounded geometry with a common geometric bound; i.e., their disjoint union is of bounded geometry.

[1] Kronecker's delta is used here.

Chapter 5

A universal foliated space

This chapter contains the proofs of the results about $\mathcal{M}_*^\infty$ stated in Section 1.3.

5.1 Quasi-isometries

Let $\phi\colon M \to N$ be a C^1 map between Riemannian manifolds. Recall that ϕ is called a (λ-) *quasi-isometry*, or (λ-) *quasi-isometric*, if there is some $\lambda \geq 1$ such that $\frac{1}{\lambda}|\xi| \leq |\phi_*(\xi)| \leq \lambda|\xi|$ for every $\xi \in TM$. This λ is called a *dilation bound* of ϕ. The second of the above inequalities, $|\phi_*(\xi)| \leq \lambda|\xi|$ for all $\xi \in TM$, means that $|\phi_*| \leq \lambda$; i.e., $|\phi_{*x}| \leq \lambda$ for all $x \in M$.

Remark 5.1.1. (i) Every quasi-isometry is an immersion.

(ii) If $|\phi_*| \leq \lambda$, then ϕ is λ-Lipschitz; i.e., $d_N(\phi(x), \phi(y)) \leq \lambda\, d_M(x,y)$ for all $x, y \in M$.

(iii) If $\phi : M \to N$ is a λ-quasi-isometry, then ϕ is λ-bi-Lipschitz; i.e., for all $x, y \in M$,

$$\frac{1}{\lambda}\, d_M(x,y) \leq d_N(\phi(x), \phi(y)) \leq \lambda\, d_M(x,y)\ .$$

(iv) Let $\psi : N \to L$ be another C^1 map between Riemannian manifolds. If $|\phi_*| \leq \lambda$ and $|\psi_*| \leq \mu$, then $|(\psi\phi)_*| \leq \lambda\mu$.

(v) The composition of a λ-quasi-isometry and a μ-quasi-isometry is a $\lambda\mu$-quasi-isometry.

(vi) The inverse of a λ-quasi-isometric diffeomorphism is a λ-quasi-isometric diffeomorphism.

Consider the subbundle $T^{\leq r}M = \{\,\xi \in TM \mid |\xi| \leq r\,\} \subset TM$ for each $r > 0$. If M has no boundary, then $T^{\leq r}M$ is a manifold with boundary, being $\partial T^{\leq r}M = T^rM := \{\,\xi \in TM \mid |\xi| = r\,\}$; otherwise, $T^{\leq r}M$ is a manifold with corners. Also, define $T^{(m),\leq r}M$ by induction on $m \in \mathbb{Z}^+$, setting $T^{(1),\leq r}M = T^{\leq r}M$ and $T^{(m),\leq r}M = T^{\leq r}T^{(m-1),\leq r}M$. Note that $T^{(m),\leq r}T^{(m'),\leq r}M = T^{(m+m'),\leq r}M$.

Definition 5.1.1. (i) It is said that $\phi : M \to N$ is a (λ-) *quasi-isometry of order* $m \in \mathbb{N}$, or a (λ-) *quasi-isometric map of order* m, if it is C^{m+1} and $\phi_*^{(m)} : T^{(m),\leq 1}M \to T^{(m)}N$ is a (λ-) quasi-isometry. This λ is called a *dilation bound of order* m of ϕ. The infimum of all dilations bounds of order m is called the *dilation of order* m. If ϕ is a quasi-isometry of order m for all $m \in \mathbb{N}$, then it is called a *quasi-isometry of order* ∞.

(ii) A collection Φ of maps between Riemannian manifolds is called a family of *equi-quasi-isometries of order* $m \in \mathbb{N}$ if it is a family of quasi-isometries of order m with some common dilation bound of order m, which is called an *equi-dilation bound of order* m. If Φ is a collection of equi-quasi-isometries of order m for all $m \in \mathbb{N}$, then it is called a family of *equi-quasi-isometries of order* ∞.

(iii) A Riemannian manifold M is said to be *quasi-isometric with order* m to another Riemannian manifold N when there is a quasi-isometric diffeomorphism of order m, $M \to N$. With more generality, a collection $\{M_i\}$ of Riemannian manifolds is called *equi-quasi-isometric with order* m to another collection $\{N_i\}$ of Riemannian manifolds, with the same index set, when there is a collection of equi-quasi-isometric diffeomorphisms of order m, $\{M_i \to N_i\}$.

Remark 5.1.2. (i) The λ-quasi-isometries of order 0 are the λ-quasi-isometries.

(ii) By Remark 4.2.1-(i), if ϕ is a λ-quasi-isometry of order $m \geq 1$, then it is a λ-quasi-isometry of order $m-1$.

(iii) For integers $0 \leq m' \leq m$, if ϕ is a λ-quasi-isometry of order m, then $\phi_*^{(m')}$ is a λ-quasi-isometry of order $m - m'$.

To begin with, let us clarify the concept of quasi-isometry of order 1. Consider the splittings $T^{(2)}M = \mathcal{H} \oplus \mathcal{V}$ and $T^{(2)}N = \mathcal{H}' \oplus \mathcal{V}'$, where $\mathcal{H}$ and $\mathcal{H}'$ are the horizontal subbundles, and $\mathcal{V}$ and $\mathcal{V}'$ are the vertical subbundles. Fix any $x \in M$ and $\xi \in T_xM$, and let $x' = \phi(x)$ and $\xi' = \phi_*(\xi)$. We have the canonical identities

$$T_\xi TM = \mathcal{H}_\xi \oplus \mathcal{V}_\xi \equiv T_xM \oplus T_xM \ , \quad T_{\xi'}TN = \mathcal{H}'_{\xi'} \oplus \mathcal{V}'_{\xi'} \equiv T_{x'}N \oplus T_{x'}N \ . \tag{5.1}$$

The pull-back Riemannian vector bundle ϕ^*TN is endowed with the pull-back ∇' of the Riemannian connection of N, and let $\phi_* : TM \to \phi^*TN$ also denote the homomorphism over id_M induced by ϕ. Let X be a C^∞ tangent vector field on some neighborhood of x in M so that $X(x) = \xi$; thus ϕ_*X is a C^1 local section of ϕ^*TN around x satisfying

$(\phi_* X)(x) = \xi' \in (\phi^* TN)_x \equiv T_{\phi(x)}N$. Then, for any $\zeta \in T_x M$ and each C^∞ function f defined on some neighborhood of x, we have

$$\begin{aligned}\nabla'_\zeta(\phi_*(fX)) - \phi_*(\nabla_\zeta(fX)) &= f(x)\,\nabla'_\zeta(\phi_* X) + df(\zeta)\,\phi_*\xi - f(x)\,\phi_*(\nabla_\zeta X) - df(\zeta)\,\phi_*\xi \\ &= f(x)\,(\nabla'_\zeta(\phi_* X) - \phi_*(\nabla_\zeta X))\end{aligned}$$

in $(\phi^* TN)_x \equiv T_{x'}N$. Therefore $A_\phi(\zeta \otimes \xi) := \nabla'_\zeta(\phi_* X) - \phi_*(\nabla_\zeta X)$ depends only on $\zeta \otimes \xi$, and this expression defines a continuous section A_ϕ of $TM^* \otimes TM^* \otimes \phi^* TN$. Observe that X can be chosen so that $\nabla_\zeta X = 0$, giving $A_\phi(\zeta \otimes \xi) = \nabla'_\zeta(\phi_* X)$ in this case. Then, from the definitions of tangent map and covariant derivative, it easily follows that, according to (5.1),

$$\phi_{**\xi}(\zeta_1, \zeta_2) \equiv (\phi_*(\zeta_1), \phi_*(\zeta_2) + A_\phi(\zeta_1 \otimes \xi)) \tag{5.2}$$

for all $\zeta_1, \zeta_2 \in T_x M$.

Remark 5.1.3. If TM were used instead of $T^{\leq 1}M$ in the definition of quasi-isometries of order 1, we would get $A_\phi = 0$, which is too restrictive. On the other hand, it would be weaker to use $T^1 M$ instead of $T^{\leq 1}M$.

Lemma 5.1.2. *Suppose that $\phi : M \to N$ is C^2. Then the following properties hold for $r > 0$ and $\mu, \nu, K \geq 0$:*

(i) *If $|\phi_{**\xi}| \leq \mu$ for all $\xi \in T^{\leq r}M$, then $|\phi_*| \leq \mu$ and $|A_\phi| \leq \mu/r$.*

(ii) *If $|\phi_*| \leq \nu$ and $|A_\phi| \leq K$, then $|\phi_{**\xi}| \leq \sqrt{2}(\nu + Kr)$ for all $\xi \in T^{\leq r}M$.*

Proof. Assume that $|\phi_{**\xi}| \leq \mu$ for all $\xi \in T^{\leq r}M$. We get $|\phi_*| \leq \mu$ by Remark 4.2.1-(i). Furthermore, for all $x \in M$ and $\xi, \zeta \in T_x M$ with $|\xi| = r$, according to (5.1) and (5.2),

$$|A_\phi(\zeta \otimes \xi)| \leq |(\phi_{*x}(\zeta), A_\phi(\zeta \otimes \xi))| = |\phi_{**\xi}(\zeta, 0)| \leq \mu\,|(\zeta, 0)| = \mu\,|\zeta| = \frac{\mu}{r}\,|\zeta|\,|\xi|\,.$$

Now, suppose that $|\phi_*| \leq \nu$ and $|A_\phi| \leq K$. Fix all $x \in M$ and $\xi, \zeta_1, \zeta_2 \in T_x M$ with $|\xi| \leq r$, according to (5.1) and (5.2),

$$\begin{aligned}|\phi_{**\xi}(\zeta_1, \zeta_2)| &\leq |\phi_*(\zeta_1)| + |\phi_*(\zeta_2) + A_\phi(\zeta_1 \otimes \xi)| \leq \nu\,|\zeta_1| + \nu\,|\zeta_2| + K\,|\zeta_1|\,|\xi| \\ &\leq \nu\,|\zeta_1| + \nu\,|\zeta_2| + Kr\,|\zeta_1| \leq (\nu + Kr)\,(|\zeta_1| + |\zeta_2|) \leq \sqrt{2}(\nu + Kr)\,|(\zeta_1, \zeta_2)|\,. \quad \square\end{aligned}$$

Lemma 5.1.3. *Suppose that $\phi : M \to N$ is C^2. Then the following conditions are equivalent for $r > 0$:*

(i) *$\phi_* : T^{\leq r}M \to TN$ is a quasi-isometry.*

(ii) *ϕ is a quasi-isometry and $|A_\phi|$ is uniformly bounded.*

In this case, the constants involved in the above properties are related in the following way:

(a) *If μ is a dilation bound of $\phi_* : T^{\leq r}M \to TN$, then μ is a dilation bound of ϕ and $|A_\phi| \leq \mu/r$.*

(b) *If ν is a dilation bound of ϕ, $|A_\phi| \leq K$, and $0 < \kappa < 1$ with $\nu K \kappa r < 1$, then*

$$\mu = \max\left\{\sqrt{2}(\nu + Kr), \frac{\sqrt{2}\nu}{1 - \nu K \kappa r}, \frac{\sqrt{2}\nu}{\kappa}\right\}$$

is a dilation bound of $\phi_* : T^{\leq r}M \to TN$.

Proof. Assume that ((i)) holds, and let μ be a dilation bound of order 1 of ϕ. Then ϕ is a μ-quasi-isometry by Remark 4.2.1-(i). This shows (ii) and (a) by Lemma 5.1.2-(i).

Now, suppose that (ii) holds, and take ν, K, κ and μ like in (b). For all $x \in M$ and $\xi, \zeta_1, \zeta_2 \in T_xM$ with $|\xi| \leq r$, according to (5.1) and (5.2),

$$\begin{aligned}|\phi_{**\xi}(\zeta_1, \zeta_2)| &\geq \frac{1}{\sqrt{2}}\left(|\phi_*(\zeta_1)| + |\phi_*(\zeta_2) + A_\phi(\zeta_1 \otimes \xi)|\right) \geq \frac{1}{\sqrt{2}}\left(|\phi_*(\zeta_1)| + \kappa\,|\phi_*(\zeta_2) + A_\phi(\zeta_1 \otimes \xi)|\right)\\ &\geq \frac{1}{\sqrt{2}}\left(|\phi_*(\zeta_1)| + \kappa(|\phi_*(\zeta_2)| - |A_\phi(\zeta_1 \otimes \xi)|)\right) \geq \frac{1}{\sqrt{2}}\left(\left(\frac{1}{\nu} - K\kappa\,|\xi|\right)|\zeta_1| + \frac{\kappa}{\nu}\,|\zeta_2|\right)\\ &\geq \frac{1}{\sqrt{2}}\left(\left(\frac{1}{\nu} - K\kappa r\right)|\zeta_1| + \frac{\kappa}{\nu}\,|\zeta_2|\right) \geq \frac{1}{\mu}\left(|\zeta_1| + |\zeta_2|\right) \geq \frac{1}{\mu}\,|(\zeta_1, \zeta_2)|\,.\end{aligned}$$

This gives ((i)) and (b) by Lemma 5.1.2-(ii). □

For $c > 0$, let $h_c : TM \to TM$ be the C^∞ diffeomorphism defined by $h_c(\xi) = c\xi$. Observe that $h_c(T^{\leq 1}M) = T^{\leq c}M$, and the following diagram is commutative:

$$\begin{tikzcd} TM \arrow[r, "\phi_*"] \arrow[d, "h_c"'] & TN \arrow[d, "h_c"] \\ TM \arrow[r, "\phi_*"] & TN \end{tikzcd}$$

For each $m \in \mathbb{Z}^+$, let $\mathcal{H}^{(m+1)}$ and $\mathcal{V}^{(m+1)}$ denote the horizontal and vertical vector subbundles of $T^{(m+1)}M$ over $T^{(m)}M$. Thus, for $\xi \in T^{(m-1)}M$ and $\zeta \in T_\xi T^{(m-1)}M$,

$$T_\zeta T^{(m)}M = \mathcal{H}^{(m+1)}_\zeta \oplus \mathcal{V}^{(m+1)}_\zeta \equiv T_\xi T^{(m-1)}M \oplus T_\xi T^{(m-1)}M\,. \tag{5.3}$$

Lemma 5.1.4. *For all $m \in \mathbb{Z}^+$, there exists an orthogonal vector bundle decomposition, $T^{(m+1)}M = \mathcal{P}^{(m+1)} \oplus \mathcal{Q}^{(m+1)}$, preserved by $h^{(m)}_{c*}$, such that, for $\xi \in T^{(m-1)}M$, $\zeta \in T_\xi T^{(m-1)}M$ and $\zeta' = h^{(m)}_{c*}(\zeta)$, the canonical identity $T_\zeta T^{(m)}M \equiv T_{\zeta'}T^{(m)}M$ given by (5.3) induces identities, $\mathcal{P}^{(m+1)}_\zeta \equiv \mathcal{P}^{(m+1)}_{\zeta'}$ and $\mathcal{Q}^{(m+1)}_\zeta \equiv \mathcal{Q}^{(m+1)}_{\zeta'}$, so that $h^{(m)}_{c*} : \mathcal{P}^{(m+1)}_\zeta \to \mathcal{P}^{(m+1)}_{\zeta'} \equiv \mathcal{P}^{(m+1)}_\zeta$ is the identity, and $h^{(m)}_{c*} : \mathcal{Q}^{(m+1)}_\zeta \to \mathcal{Q}^{(m+1)}_{\zeta'} \equiv \mathcal{Q}^{(m+1)}_\zeta$ is multiplication by c.*

Proof. The proof is by induction on m. By the definition of connection, h_{c*} preserves the orthogonal decomposition $T^{(2)}M = \mathcal{H} \oplus \mathcal{V}$. Moreover, for $\zeta \in TM$ and $\zeta' = c\zeta$, $h_{c*} : \mathcal{H}_\zeta \to \mathcal{H}_{\zeta'} \equiv \mathcal{H}_\zeta$ is the identity, and $h_{c*} : \mathcal{V}_\zeta \to \mathcal{V}_{\zeta'} \equiv \mathcal{V}_\zeta$ is multiplication by c. Thus the statement is true in this case with $\mathcal{P}^{(2)} = \mathcal{H}$ and $\mathcal{Q}^{(2)} = \mathcal{V}$.

Now, suppose that $m \geq 2$ and the result holds for $m-1$. For $\xi \in T^{(m-1)}M$ and $\zeta \in T_\xi T^{(m-1)}M$, we have canonical identities

$$\mathcal{H}_\zeta^{(m+1)} \equiv \mathcal{V}_\zeta^{(m+1)} \equiv T_\xi T^{(m-1)}M = \mathcal{P}_\xi^{(m)} \oplus \mathcal{Q}_\xi^{(m)} \,, \tag{5.4}$$

obtaining orthogonal decompositions, $\mathcal{H}^{(m+1)} = \mathcal{H}\mathcal{P}^{(m)} \oplus \mathcal{H}\mathcal{Q}^{(m)}$ and $\mathcal{V}^{(m+1)} = \mathcal{V}\mathcal{P}^{(m)} \oplus \mathcal{V}\mathcal{Q}^{(m)}$, where $(\mathcal{H}\mathcal{P}^{(m)})_\zeta \equiv \mathcal{P}_\xi^{(m)} \equiv (\mathcal{V}\mathcal{P}^{(m)})_\zeta$ and $(\mathcal{H}\mathcal{Q}^{(m)})_\zeta \equiv \mathcal{Q}_\xi^{(m)} \equiv (\mathcal{V}\mathcal{Q}^{(m)})_\zeta$ according to (5.4). Then the result follows with $\mathcal{P}^{(m+1)} = \mathcal{H}\mathcal{P}^{(m)} \oplus \mathcal{V}\mathcal{P}^{(m)}$ and $\mathcal{Q}^{(m+1)} = \mathcal{H}\mathcal{Q}^{(m)} \oplus \mathcal{V}\mathcal{Q}^{(m)}$. □

Corollary 5.1.5. *For all $m \in \mathbb{Z}^+$ and $c, r > 0$, we have $h_{c*}^{(m)}(T^{(m+1),\leq r}M) \subset T^{(m+1),\leq \bar{c}r}M$, where $\bar{c} = \max\{c, 1\}$, and $h_{c*}^{(m)} : T^{(m+1)}M \to T^{(m+1)}M$ is a $\hat{c}$-quasi-isometry, where $\hat{c} = \max\{c, 1/c\}$.*

Lemma 5.1.6. *For all $m \in \mathbb{Z}^+$, $r, s > 0$ and $\lambda \geq 0$, there is some $\mu \geq 0$ such that, for any C^{m+1} map between Riemannian manifolds, $\phi : M \to N$, if $|(\phi_*^{(m)})_{*\xi}| \leq \lambda$ for all $\xi \in T^{(m),\leq r}M$, then $|(\phi_*^{(m)})_{*\xi}| \leq \mu$ for all $\xi \in T^{(m),\leq s}M$. Moreover μ can be chosen so that $\mu s \to 0$ as $s \to 0$ for fixed m, r and λ.*

Proof. We proceed by induction on m.

For $m = 1$, we have $|\phi_{**\xi}| \leq \lambda$ for all $\xi \in T^{\leq r}M$. Then $|\phi_*| \leq \lambda$ and $|A_\phi| \leq \lambda/r$ by Lemma 5.1.2-(i). Using Lemma 5.1.2-(ii), it follows that $|\phi_{**\xi}| \leq \sqrt{2}\lambda(1 + s/r) =: \mu$ for all $\xi \in T^{\leq s}M$. Note that $\mu s \to 0$ as $s \to 0$ for fixed r and λ in this case.

Now, assume that $m \geq 2$ and the result holds for $m-1$. For $c = r/s$ and $t = \min\{cr, r\}$, the diagram

$$\begin{tikzcd} T^{(m),\leq r}M \arrow[r, "\phi_*^{(m)}"] & T^{(m)}N \arrow[d, "h_{c*}^{(m-1)}"] \\ T^{(m-1),\leq t}T^{\leq s}M \arrow[u, "h_{1/c*}^{(m-1)}"] \arrow[r, "\phi_*^{(m)}"] & T^{(m)}N \end{tikzcd} \tag{5.5}$$

is defined and commutative. Using Corollary 5.1.5 and Remark 5.1.1-(iv), we obtain $|(\phi_*^{(m)})_{*\xi}| \leq \hat{c}^2\lambda$ for all $\xi \in T^{(m-1),\leq t}T^{\leq s}M$, where $\hat{c} = \max\{c, 1/c\}$. Then, by the induction hypothesis applied to the map $\phi_* : T^{\leq s}M \to TN$, there is some $\mu \geq 0$, depending only on $m-1$, t, s and $\hat{c}^2\lambda$, such that $|(\phi_*^{(m)})_{*\xi}| \leq \mu$ for all $\xi \in T^{(m-1),\leq s}T^{\leq s}M = T^{(m),\leq s}M$, and so that $\mu s \to 0$ as $s \to 0$ for fixed m, t and $\hat{c}^2\lambda$. □

Corollary 5.1.7. *For all $m \in \mathbb{Z}^+$, $r > 0$ and $\lambda \geq 0$, there is some $s > 0$ such that, for any C^{m+1} map between Riemannian manifolds, $\phi : M \to N$, if $|(\phi_*^{(m)})_{*\xi}| \leq \lambda$ for all $\xi \in T^{(m),\leq 1}M$, then $\phi_*^{(m+1)}(T^{(m+1),\leq s}M) \subset T^{(m+1),\leq r}N$.*

Proof. This is also proved by induction on m. The statement is true for $m = 0$ because, if $|\phi_*| \leq \lambda$, then $\phi_*(T^{\leq s}M) \subset T^{\leq \lambda s}N$ for all $s > 0$, and therefore it is enough to take $s = r/\lambda$ in this case.

Now, assume that $m \geq 1$ and the result is true for $m - 1$. By Remark 4.2.1-(i), if $|(\phi_*^{(m)})_{*\xi}| \leq \lambda$ for all $\xi \in T^{(m),\leq 1}M$, then $|(\phi_*^{(m-1)})_{*\xi}| \leq \lambda$ for all $\xi \in T^{(m-1),\leq 1}M$. Hence, by the induction hypothesis, for all $r > 0$, there is some $s > 0$, as small as desired, such that $\phi_*^{(m)}(T^{(m),\leq s}M) \subset T^{(m),\leq r}N$. On the other hand, by Lemma 5.1.6, there is some $\mu > 0$, depending on m, r, s and λ, such that $|(\phi_*^{(m)})_{*\xi}| \leq \mu$ for all $\xi \in T^{(m),\leq s}M$, and satisfying $\mu s \to 0$ as $s \to 0$ for fixed m, r and λ. Thus we can choose s, and the corresponding μ, so that $\mu s \leq r$. Then

$$\phi_*^{(m+1)}(T^{(m+1),\leq s}M) \subset T^{\leq \mu s}T^{(m),\leq r}N \subset T^{(m+1),\leq r}N\,. \quad \square$$

Lemma 5.1.8. *For $m \in \mathbb{Z}^+$, $r, s > 0$ and $\lambda \geq 1$, there is some $\mu \geq 1$ such that, for any C^{m+1} map between Riemannian manifolds, $\phi : M \to N$, if $\phi_*^{(m)} : T^{(m),\leq r}M \to T^{(m)}N$ is a λ-quasi-isometry, then $\phi_*^{(m)} : T^{\leq s}M \to T^{(m)}N$ is a μ-quasi-isometry.*

Proof. Again, we use induction on m. The case $m = 1$ is a direct consequence of Lemma 5.1.3.

Now, assume that $m \geq 2$ and the result holds for $m - 1$. Consider the notation of the proof of Lemma 5.1.6. From the commutativity of (5.5), and using Corollary 5.1.5 and Remark 5.1.1-(v), it follows that the lower horizontal arrow of (5.5) is a $\hat{c}^2\lambda$-quasi-isometry. Then, by the induction hypothesis applied to the map $\phi_* : T^{\leq s}M \to TN$, there is some $\mu > 0$, depending only on $m-1$, t, s and $\hat{c}^2\lambda$, such that $\phi_*^{(m)} : T^{(m),\leq s}M \to T^{(m)}N$ is a μ-quasi-isometry. $\square$

Remark 5.1.4. According to Lemma 5.1.8, we could use any $T^{(m),\leq r}M$ instead of $T^{(m),\leq 1}M$ to define quasi-isometries of order m, but the dilation bounds of order m would be different.

Proposition 5.1.9. (i) *For all $m \in \mathbb{N}$ and $\lambda, \mu \geq 1$, there is some $\nu \geq 1$ such that, if $\phi : M \to N$ and $\psi : N \to L$ are quasi-isometries of order m, and λ and μ are respective dilation bounds of order m, then $\psi\phi$ is a ν-quasi-isometry of order m.*

(ii) *For all $m \in \mathbb{N}$ and $\lambda \geq 1$, there is some $\mu \geq 1$ such that, if $\phi : M \to N$ is a λ-quasi-isometric diffeomorphism of order m, then ϕ^{-1} is a μ-quasi-isometry of order m.*

Proof. Let us prove (i). By Corollary 5.1.7, there is some $r > 0$, depending on m and λ, such that

$$\phi_*^{(m+1)}(T^{(m+1),\leq r}M) \subset T^{(m+1),\leq 1}N ,$$

and therefore $\phi_*^{(m)}(T^{(m),\leq r}M) \subset T^{(m),\leq 1}N$. On the other hand, by Lemma 5.1.8, there is some $\lambda' \geq 1$, depending on m, r and λ, such that $\phi_*^{(m)} : T^{(m),\leq r}M \to T^{(m),\leq 1}N$ is a λ'-quasi-isometry. So

$$(\psi\phi)_*^{(m)} = \psi_*^{(m)}\phi_*^{(m)} : T^{(m),\leq r}M \to T^{(m)}L$$

is a $\lambda'\mu$-quasi-isometry by Remark 5.1.1-(v). Thus, by Lemma 5.1.8, there is some $\nu \geq 1$, depending on m, r and $\lambda'\mu$, so that $(\psi\phi)_*^{(m)} : T^{(m),\leq 1}M \to T^{(m)}L$ is a ν-quasi-isometry; i.e., $\psi\phi$ is a ν-quasi-isometry of order m.

Now, let us prove (ii). By Corollary 5.1.7, there is some $r > 0$, depending on m and λ, such that

$$(\phi^{-1})_*^{(m+1)}(T^{(m+1),\leq r}N) \subset T^{(m+1),\leq 1}M ,$$

and therefore $(\phi^{-1})_*^{(m)}(T^{(m),\leq r}N) \subset T^{(m),\leq 1}M$. So

$$\phi_*^{(m)} : (\phi^{-1})_*^{(m)}(T^{(m),\leq r}N) \to T^{(m),\leq r}N$$

is a λ-quasi-isometric diffeomorphism, obtaining that

$$(\phi^{-1})_*^{(m)} = (\phi_*^{(m)})^{-1} : T^{(m),\leq r}N \to (\phi^{-1})_*^{(m)}(T^{(m),\leq r}N)$$

is a λ-quasi-isometry by Remark 5.1.1-(vi). Thus, by Lemma 5.1.8, there is some $\mu \geq 1$, depending on m, r and λ, so that $(\phi^{-1})_*^{(m)} : T^{(m),\leq 1}N \to T^{(m)}M$ is a μ-quasi-isometry; i.e., ϕ^{-1} is a μ-quasi-isometry of order m. □

Corollary 5.1.10. *"Being quasi-isometric with order m" is an equivalence relation.*

Let M and N be connected Riemannian manifolds. For every $m \in \mathbb{N} \cup \{\infty\}$, consider the weak C^m topology on $C^m(M, N)$ (see [44]). For $x \in M$ and $\Phi \subset C^m(M, N)$, let $\Phi(x) = \{\, \phi(x) \mid \phi \in \Phi \,\} \subset N$.

Proposition 5.1.11. *Assume that N is complete. Let $x_0 \in M$, and let $\Phi \subset C^{m+1}(M, N)$ be a family of equi-quasi-isometries of order $m \in \mathbb{N} \cup \{\infty\}$. Then Φ is precompact in $C^m(M, N)$ if and only if $\Phi(x_0)$ is bounded in N.*

Proof. The "only if" part follows because the evaluation map $C^m(M, N) \to N$, $\phi \mapsto \phi(x_0)$, is continuous.

For $m \in \mathbb{N}$, the "if" part is proved by induction. For $m = 0$, the assumption that $\Phi \subset C^1(M, N)$ is a family of equi-quasi-isometries implies that Φ is equi-continuous by

Remark 5.1.1-(iii). On the other hand, $\Phi(x) \subset \mathrm{Pen}_N(\Phi(x_0), \lambda\, d(x, x_0))$ for any $x \in M$ by Remark 5.1.1-(iii), where $\lambda \geq 1$ is an equi-dilation bound of Φ. So $\Phi(x)$ is precompact in N because $\Phi(x_0)$ is bounded and N is complete. Therefore Φ is precompact in $C(M, N)$ by the Arzelà-Ascoli theorem.

Now, take an integer $m \geq 1$ and assume that the result holds for $m - 1$. The map $C^m(M, N) \to C^{m-1}(T^{\leq 1}M, TN)$, $\phi \mapsto \phi_*|_{T^{\leq 1}M}$, is an embedding. So it is enough to prove that the image Φ_* of Φ by this map is precompact in $C^{m-1}(T^{\leq 1}M, TN)$. This holds by the induction hypothesis because $\Phi_* \subset C^m(T^{\leq 1}M, TN)$ is a family of equi-quasi-isometries of order $m - 1$ by Remark 5.1.2-(iii).

The "if" part for $m = \infty$ can be proved as follows. In this case, we have proved that Φ is precompact in $C^l(M, N)$ for every $l \in \mathbb{N}$. By the continuity of the inclusion maps $C^{l+1}(M, N) \hookrightarrow C^l(M, N)$, it follows that Φ has the same closure $\overline{\Phi}$ in $C^l(M, N)$ and $C^{l+1}(M, N)$, and the weak C^l and C^{l+1} topologies coincide on $\overline{\Phi}$. Therefore $\overline{\Phi}$ is the closure of Φ in $C^\infty(M, N)$ too, and the weak C^∞ and C^l topologies coincide on $\overline{\Phi}$ for any $l \in \mathbb{N}$. Thus Φ is precompact in $C^\infty(M, N)$. □

5.2 Partial quasi-isometries

Let M and N be connected complete Riemannian manifolds without boundary.

Definition 5.2.1. For $m \in \mathbb{N}$, a partial map $f : M \rightarrowtail N$ is called a C^m *local diffeomorphism* if $\operatorname{dom} f$ and $\operatorname{im} f$ are open in M and N, respectively, and $f : \operatorname{dom} f \to \operatorname{im} f$ is a C^m diffeomorphism. If moreover $f(x) = y$ for distinguished points, $x \in \operatorname{dom} f$ and $y \in \operatorname{im} f$, then f is said to be *pointed*, and the notation $f : (M, x) \rightarrowtail (N, y)$ is used. The term *local homeomorphism* is used in the C^0 case.

The term "C^m local diffeomorfism" ($m \geq 1$) may be also used in the standard sense, referring to any C^m map $M \to N$ whose tangent map is an isomorphism at every point of M. The context will always clarify this ambiguity.

Definition 5.2.2. For $m \in \mathbb{N}$, $R > 0$ and $\lambda \geq 1$, a C^{m+1} pointed local diffeomorphism $\phi\colon (M, x) \rightarrowtail (N, y)$ is called an (m, R, λ)*-pointed local quasi-isometry*, or a *local quasi-isometry* of *type* (m, R, λ), if the restriction $\phi_*^{(m)} : \Omega^{(m)} \to T^{(m)}N$ is a λ-quasi-isometry for some compact domain $\Omega^{(m)} \subset \operatorname{dom} \phi_*^{(m)}$ with $B_M^{(m)}(x, R) \subset \Omega^{(m)}$.

Remark 5.2.1. (i) Any pointed local quasi-isometry $(M, x) \rightarrowtail (N, y)$ of type (m, R, λ) is also of type (m', R', λ') for $0 \leq m' \leq m$, $0 < R' < R$ and $\lambda' > \lambda$ (using Remark 4.2.1-(i)).

(ii) For integers $0 \le m' \le m$, any pointed C^{m+1} local diffeomorphism $\phi : (M,x) \rightarrowtail (N,y)$ is a pointed local quasi-isometry of type (m,R,λ) if and only if $\phi_*^{(m')} : (T^{(m')}M,x) \rightarrowtail (T^{(m')}N,y)$ is a pointed local quasi-isometry of type $(m-m',R,\lambda)$.

(iii) If there is an (m,R,λ)-pointed local quasi-isometry $(M,x) \rightarrowtail (N,y)$, then, for all $R' < R$ and $\lambda' > \lambda$, there is a C^∞ (m,R',λ')-pointed local quasi-isometry $(M,x) \rightarrowtail (N,y)$ by [44, Theorem 2.7].

Lemma 5.2.3. *The following properties hold:*

(i) *If $\phi : (M,x) \rightarrowtail (N,y)$ and $\psi : (N,y) \rightarrowtail (L,z)$ are pointed local quasi-isometries of types (m,R,λ) and $(m,\lambda R,\lambda')$, respectively, then $\psi \circ \phi : (M,x) \rightarrowtail (L,z)$ is an $(m,R,\lambda\lambda')$-pointed local quasi-isometry.*

(ii) *If $\phi : (M,x) \rightarrowtail (N,y)$ is an $(m,\lambda R,\lambda)$-pointed local quasi-isometry, then $\phi^{-1} : (N,y) \rightarrowtail (M,x)$ is an (m,R,λ)-pointed local quasi-isometry.*

Proof. To prove (i), it is enough to show that $\overline{B}_M^{(m)}(x,R) \subset \operatorname{dom}(\psi\circ\phi)_*^{(m)}$ by Remark 5.1.1-(v). For $\xi \in \overline{B}_M^{(m)}(x,R)$, we have $\xi \in \operatorname{dom}\phi$ and $d_N^{(m)}(y,\phi_*^{(m)}(\xi)) \le \lambda\, d_M^{(m)}(x,\xi) \le \lambda R$ by Remark 5.1.1-(iii), obtaining that $\xi \in \operatorname{dom}(\psi\circ\phi)_*^{(m)}$ since $(\psi\circ\phi)_*^{(m)} = \psi_*^{(m)} \circ \phi_*^{(m)}$.

To prove (ii), it is enough to show that $\overline{B}_N^{(m)}(y,R) \subset \phi_*^{(m)}(\overline{B}_M^{(m)}(x,\lambda R))$ by Remark 5.1.1-(vi). Let $A = \overline{B}_N^{(m)}(y,R) \cap \operatorname{im}\phi_*^{(m)}$, which is open in $\overline{B}_N^{(m)}(y,R)$ and contains y. For any $\zeta \in A$, there is some $\xi \in \operatorname{dom}\phi_*^{(m)}$ so that $\phi_*^{(m)}(\xi) = \zeta$. Then $d_M^{(m)}(x,\xi) \le \lambda d_N(y,\zeta) \le \lambda R$ by Remark 5.1.1-(iii), obtaining that $\xi \in \overline{B}_M^{(m)}(x,\lambda R)$. Thus $A = \phi_*^{(m)}(\overline{B}_M^{(m)}(x,\lambda R)) \cap \overline{B}_N^{(m)}(y,R)$, which is closed in $\overline{B}_N^{(m)}(y,R)$. Therefore $\overline{B}_N^{(m)}(y,R) = A \subset \phi_*^{(m)}(\overline{B}_M^{(m)}(x,\lambda R))$ because $\overline{B}_N^{(m)}(y,R)$ is connected. □

5.3 The C^∞ topology on $\mathcal{M}_*(n)$

Definition 5.3.1. For $m \in \mathbb{N}$ and $R,r > 0$, let $U_{R,r}^m$ be the set of pairs $([M,x],[N,y]) \in \mathcal{M}_*(n) \times \mathcal{M}_*(n)$ such that there is some (m,R,λ)-pointed local quasi-isometry $(M,x) \rightarrowtail (N,y)$ for some $\lambda \in [1,e^r)$.

The following standard notation is used for a set X and relations $U,V \subset X \times X$:

$$U^{-1} = \{\, (y,x) \in X \times X \mid (x,y) \in U \,\}\,,$$
$$V \circ U = \{\, (x,z) \in X \times X \mid \exists y \in X \text{ so that } (x,y) \in U \text{ and } (y,z) \in V \,\}\,.$$

Moreover the diagonal of $X \times X$ is denoted by Δ.

Proposition 5.3.2. *The following properties hold for all $m,m' \in \mathbb{N}$ and $R,S,r,s > 0$:*

(i) $(U^m_{e^r R,r})^{-1} \subset U^m_{R,r}$.

(ii) $U^{m_0}_{R_0,r_0} \subset U^m_{R,r} \cap U^{m'}_{S,s}$, *where* $m_0 = \max\{m, m'\}$, $R_0 = \max\{R, S\}$ *and* $r_0 = \min\{r, s\}$.

(iii) $\Delta \subset U^m_{R,r}$.

(iv) $U^m_{e^s R,r} \circ U^m_{R,s} \subset U^m_{R,r+s}$.

Proof. Properties (ii) and (iii) are elementary, while Properties (i) and (iv) are consequences of Lemma 5.2.3. □

Proposition 5.3.3. $\bigcap_{R,r>0} U^m_{R,r} = \Delta$ *for all* $m \in \mathbb{N}$.

Proof. We only prove "$\subset$" because "$\supset$" is obvious. For $([M, x], [N, y]) \in \bigcap_{R,r>0} U^m_{R,r}$, there is a sequence of pointed local quasi-isometries $\phi_i : (M, x) \rightarrowtail (N, y)$, with corresponding types (m, R_i, λ_i), such that $R_i \uparrow \infty$ and $\lambda_i \downarrow 1$ as $i \to \infty$. Let us prove that $[M, x] = [N, y]$.

First, we inductively construct a pointed isometric immersion $\psi : (M, x) \to (N, y)$.

The restrictions $\phi_i : (B_M(x, R_1), x) \to (N, y)$ are pointed equi-quasi-isometries of order m (λ_1 is an equi-dilation bound of order m). By Proposition 5.1.11, there is some subsequence $\phi_{k(1,l)}$ whose restriction to $B_M(x, R_1)$ converges to some pointed C^m function $\psi_1 : (B_M(x, R_1), x) \to (N, y)$ in the weak C^m topology. Since $\lambda_i \downarrow 1$, it follows that ψ_1 is an isometric immersion.

Now assume that, for some $i \geq 1$, there is some subsequence $\phi_{k(i,l)}$ whose restriction to $B_M(x, R_i)$ converges to some pointed isometric immersion $\psi_i : (B_M(x, R_i), x) \to (N, y)$. As before, by Proposition 5.1.11, the sequence $\phi_{k(i,l)}$ has some subsequence $\phi_{k(i+1,l)}$ whose restriction to $B_M(x, R_{i+1})$ converges to some pointed isometric immersion $\psi_{i+1} : (B_M(x, R_{i+1}), x) \to (N, y)$ in the weak C^m topology. Moreover $\psi_{i+1}|_{B_M(x,R_i)} = \psi_i$. Thus the maps ψ_i can be combined to define the desired pointed isometric immersion $\psi : (M, x) \to (N, y)$.

Now, let us show that ψ is indeed a pointed isometry, and therefore $[M, x] = [N, y]$, as desired. By Lemma 5.2.3-(ii), each inverse $\phi_i^{-1} : (N, y) \rightarrowtail (M, x)$ is an (m, R'_i, λ_i)-pointed local quasi-isometry, where $R'_i = R_i/\lambda_i \uparrow \infty$. By using Proposition 5.1.11 as above, we get a subsequence $\phi^{-1}_{k'(i,l)}$ of each sequence $\phi^{-1}_{k(i,l)}$, whose restriction to $B_N(y, R'_i)$ converges to a pointed isometric immersion $\psi'_i : (B_N(y, R'_i), y) \to (M, x)$ in the weak C^m topology, and such that $\phi^{-1}_{k'(i+1,l)}$ is also a subsequence of $\phi^{-1}_{k'(i,l)}$. So $\psi'_{i+1}|_{B_N(y,R'_i)} = \psi'_i$ for all i, obtaining that the maps ψ'_i can be combined to define a pointed isometric immersion $\psi' : (N, y) \to (M, x)$. Since the operation of composition is continuous with respect to the weak C^m topology [44, p. 64, Exercise 10], we get $\psi_i \psi'_i = \mathrm{id}_{B_N(y,R'_i)}$ for all i, giving $\psi\psi' = \mathrm{id}_N$. Therefore ψ' is injective. Moreover ψ' is also surjective because M and N are complete. Hence ψ' is an isometry whose inverse is ψ. □

By Propositions 5.3.2 and 5.3.3, the sets $U^m_{R,r}$ form a base of entourages of a separating uniformity on $\mathcal{M}_*(n)$, which is called the C^∞ *uniformity*. It will be proved that the induced topology satisfies the statement of Theorem 1.3.2; thus it is called the C^∞ *topology*, and the corresponding space is denoted by $\mathcal{M}^\infty_*(n)$. The notation Cl_∞ and Int_∞ will be used for the closure and interior operators in $\mathcal{M}^\infty_*(n)$.

The following lemma will be used.

Lemma 5.3.4. *For any open $U \subset \mathcal{M}^\infty_*(n)$, the map $\mathbf{d}_U : \mathcal{M}^\infty_*(n) \to [0,\infty]$, defined by*

$$\mathbf{d}_U([M,x]) = \inf\{\, d_M(x,x') \mid x' \in M,\ [M,x'] \in U \,\} ,$$

is upper semicontinuous.

Here, recall that $\inf \emptyset = \infty$ in $\mathbb{R}$.

Proof. To prove that $\mathbf{d}_U$ is upper semicontinuous at some $[M,x] \in \mathcal{M}^\infty_*(n)$, we can assume that $D := \mathbf{d}_U([M,x]) < \infty$. Given any $\varepsilon > 0$, there is some $x' \in B_M(x, D+\varepsilon)$ such that $[M,x'] \in U$. Since U is open, we have $U^m_{R,r}(M,x') \subset U$ for some $m \in \mathbb{N}$ and $R, r > 0$ with $R \geq D + \varepsilon$ and $e^r d_M(x,x') < D + \varepsilon$. Given any $[N,y] \in U^m_{2R,r}(M,x)$, there is some $(m, 2R, \lambda)$-pointed local quasi-isometry $\phi : (M,x) \rightarrowtail (N,y)$ for some $\lambda \in [1, e^r)$. Take some $\delta > 0$ such that $\lambda(d_M(x,x') + \delta) < D + \varepsilon$, and let α be a smooth curve in $B_M(x, D+\varepsilon)$ of length $< d_M(x,x') + \delta$ from x to x'. Hence $\phi\alpha$ is a well defined C^{m+1} curve in N from y to $y' := \phi(x')$ of length $< \lambda(d_M(x,x') + \delta) < D + \varepsilon$, obtaining that $d_N(y,y') < D + \varepsilon$. On the other hand, ϕ is also an (m, R, λ)-pointed local quasi-isometry $(M,x') \rightarrowtail (N,y')$, showing that $[N,y'] \in U^m_{R,r}(M,x') \subset U$. So $\mathbf{d}_U([N,y]) < D + \varepsilon$. □

5.4 Convergence in the C^∞ topology

Lemma 5.4.1. *Let g and g' be positive definite scalar products on a real vector space V, and let $|\ |$ and $|\ |'$ denote the respective induced norms on the vector space of tensors over V. The following properties hold:*

(i) *If $\lambda \geq 1$ satisfies $\frac{1}{\lambda}|v|' \leq |v| \leq \lambda |v|'$ for all $v \in V$, then $|g - g'| \leq \lambda^2 - \lambda^{-2}$.*

(ii) *If $|g - g'| \leq \varepsilon$ for some $\varepsilon \in [0,1)$, then $\sqrt{1-\varepsilon}\,|v| \leq |v|' \leq \sqrt{1+\varepsilon}\,|v|$ for all $v \in V$.*

(iii) *If $\lambda \geq 1$ satisfies $\frac{1}{\lambda}|v|' \leq |v| \leq \lambda |v|'$ for all $v \in V$, then $\frac{1}{\lambda^2}|\omega|' \leq |\omega| \leq \lambda^2 |\omega|'$ for all $\omega \in V^* \otimes V^*$.*

Proof. To prove (i), take arbitrary vectors $v, w \in V$ with $|v| = |w| = 1$. By polarization,

$$\begin{aligned}(g-g')(v,w) &= \frac{1}{4}\left(|v+w|^2 - |v-w|^2 - |v+w|'^2 + |v-w|'^2\right)\\ &\le \frac{1}{4}\left(\left(1-1/\lambda^2\right)|v+w|^2 + (\lambda^2-1)|v-w|^2\right) \le 1 - \frac{1}{\lambda^2} + \lambda^2 - 1 = \lambda^2 - \frac{1}{\lambda^2}\,.\end{aligned}$$

Interchanging g and g' in these inequalities, it also follows that $|(g-g')(v,w)| \le \lambda^2 - \lambda^{-2}$.

Property (ii) follows because, for any $v \in V$,

$$(1-\varepsilon)|v|^2 \le |v|^2 - \left||v|^2 - |v|'^2\right| \le |v|'^2 \le |v|^2 + \left||v|^2 - |v|'^2\right| \le (1+\varepsilon)|v|^2\,.$$

Let us prove (iii). For all $v, w \in V \smallsetminus \{0\}$,

$$\frac{|\omega(v,w)|}{|v|'\,|w|'} \le \lambda^2\,\frac{|\omega(v,w)|}{|v|\,|w|} \le \lambda^2\,|\omega|\,,$$

obtaining $|\omega|' \le \lambda^2|\omega|$. Interchanging the roles of $|\ |$ and $|\ |'$, we also get $|\omega| \le \lambda^2|\omega|'$. □

The following coordinate free description of C^m convergence is a direct consequence of (4.3).

Lemma 5.4.2 (Lessa [55, Lemma 7.1]). *For $m \in \mathbb{N}$, a sequence $[M_i, x_i] \in \mathcal{M}_*(n)$ is C^m convergent to $[M, x] \in \mathcal{M}_*(n)$ if and only if, for every compact domain $\Omega \subset M$ containing x, there are pointed C^{m+1} embeddings $\phi_i\colon (\Omega, x) \to (M_i, x_i)$, for i large enough, such that $\|g_M - \phi_i^* g_{M_i}\|_{C^m,\Omega,g_M} \to 0$ as $i \to \infty$.*

Definition 5.4.3. For $R, r > 0$ and $m \in \mathbb{N}$, let $D^m_{R,r}$ be the set of pairs $([M,x],[N,y]) \in \mathcal{M}_*(n) \times \mathcal{M}_*(n)$ such that there is some C^{m+1} pointed local diffeomorphism $\phi\colon (M,x) \rightarrowtail (N,y)$ so that $\|g_M - \phi^* g_N\|_{C^m,\Omega,g_M} < r$ for some compact domain $\Omega \subset \operatorname{dom}\phi$ with $B_M(x,R) \subset \Omega$.

Given a set X, for $U \subset X \times X$ and $x \in X$, let $U(x) = \{\, y \in Y \mid (x,y) \in U \,\}$. In the case of $U \subset \mathcal{M}_*(n) \times \mathcal{M}_*(n)$ and $[M,x] \in \mathcal{M}_*(n)$, we simply write $U(M,x)$.

Remark 5.4.1. By Lemma 5.4.2, a sequence $[M_i, x_i] \in \mathcal{M}_*(n)$ is C^∞ convergent to $[M,x] \in \mathcal{M}_*(n)$ if and only if it is eventually in $D^m_{R,r}(M,x)$ for arbitrary $m \in \mathbb{N}$ and $R, r > 0$.

Proposition 5.4.4. (i) *For all $R, r > 0$, if $0 < \varepsilon \le \min\{1 - e^{-2r}, e^{2r} - 1\}$, then $D^0_{R,\varepsilon} \subset U^0_{R,r}$.*

(ii) *For all $m \in \mathbb{Z}^+$, $R, r > 0$ and $[M,x] \in \mathcal{M}_*(n)$, there is some $\varepsilon > 0$ such that $D^m_{R,\varepsilon}(M,x) \subset U^m_{R,r}(M,x)$.*

Proof. Let us show (i). If $([M,x],[N,y]) \in D^0_{R,\varepsilon}$, then there is a C^1 pointed local diffeomorphism $\phi : (M,x) \rightarrowtail (N,y)$ such that $\varepsilon_0 := \|g_M - \phi^* g_N\|_{C^0,\Omega,g_M} < \varepsilon$ for some compact domain $\Omega \subset \operatorname{dom}\phi$ with $B_M(x,R) \subset \Omega$. Choose some $\lambda \in [1,e^r)$ such that $\varepsilon_0 \le \min\{1-\lambda^{-2}, \lambda^2 - 1\}$. Set $g = g_M$ and $g' = \phi^* g_N$, and let $|\ |$ and $|\ |'$ denote the respective norms. For $\xi \in T\Omega$, we have

$$\frac{1}{\lambda}\,|\xi| \le \sqrt{1-\varepsilon_0}\,|\xi| \le |\xi|' \le \sqrt{1+\varepsilon_0}\,|\xi| \le \lambda\,|\xi|$$

by Lemma 5.4.1-(ii). Thus ϕ is a $(0,R,\lambda)$-pointed local quasi-isometry, obtaining that $([M,x],[N,y]) \in U^0_{R,r}$.

Let us prove (ii). Take $m \in \mathbb{Z}^+$, $R, r > 0$ and $[M,x] \in \mathcal{M}_*(n)$. Let $\mathcal{U}$ be a finite collection of charts of M with domains U_a, and let $\mathcal{K} = \{K_a\}$ be a family of compact subsets of M, with the same index set as $\mathcal{U}$, such that $K_a \subset U_a$ for all a, and $\overline{B}_M(x,R) \subset \operatorname{Int}(K)$ for $K = \bigcup_a K_a$. Let $\varepsilon > 0$, to be fixed later. For any $[N,y] \in D^m_{R,\varepsilon}(M,x)$, there is a C^{m+1} pointed local diffeomorphism $\phi\colon (M,x) \rightarrowtail (N,y)$ so that $\|g_M - \phi^* g_N\|_{C^m,\Omega,g_M} < \varepsilon$ for some compact domain $\Omega \subset \operatorname{dom}\phi \cap \operatorname{Int}(K)$ with $B_M(x,R) \subset \Omega$. By continuity, there is another compact domain $\Omega' \subset \operatorname{dom}\phi \cap \operatorname{Int}(K)$ such that $\Omega \subset \operatorname{Int}(\Omega')$ and $\|g_M - \phi^* g_N\|_{C^m,\Omega',g_M} < \varepsilon$. As before, let $g = g_M$ and $g' = \phi^* g_N$.

With the notation of Section 4.2, let $\mathcal{U}^{(m)}$ be the family of induced charts of $T^{(m)}M$ with domains $U^{(m)}_a$, let $\mathcal{K}^{(m)}$ be the family of compact subsets

$$K^{(m)}_a = \{\, \xi \in T^{(m)}M \mid \pi(\xi) \in K_a,\ d^{(m)}_M(\xi,\pi(\xi)) \le R' \,\} \subset U^{(m)}_a\ ,$$

for some $R' > R$, where $\pi : T^{(m)}M \to M$, and let $K^{(m)} = \bigcup_a K^{(m)}_a$. Since $\overline{B}^{(m)}_M(x,R) \subset \operatorname{Int}(K^{(m)})$ and $\pi(\overline{B}^{(m)}_M(x,R)) = \overline{B}_M(x,R) \subset \Omega \subset \operatorname{Int}(\Omega')$ by Remark 4.2.1-(iv),(v), there is some compact domain $\Omega^{(m)} \subset T^{(m)}M$ such that $B^{(m)}_M(x,R) \subset \Omega^{(m)} \subset K^{(m)}$ and $\pi(\Omega^{(m)}) \subset \Omega'$.

Choose the following constants:

- some $C \ge 1$ satisfying (4.3) with $\mathcal{U}$, $\mathcal{K}$, Ω' and g;
- some $C^{(m)} \ge 1$ satisfying (4.3) with $\mathcal{U}^{(m)}$, $\mathcal{K}^{(m)}$, $\Omega^{(m)}$ and $g^{(m)}$;
- some $\delta \in (0, \min\{1-e^{-2r}, e^{2r}-1\}]$; and,
- by Lemma 4.2.1-(i), some $\varepsilon' > 0$ such that

$$\|g - g'\|_{C^m,\Omega',\mathcal{U},\mathcal{K}} < \varepsilon' \implies \|g^{(m)} - g'^{(m)}\|_{C^0,\Omega^{(m)},\mathcal{U}^{(m)},\mathcal{K}^{(m)}} < \delta / C^{(m)}\ .$$

Suppose that $\varepsilon \leq \varepsilon'/C$. Then

$$\begin{aligned}\|g-g'\|_{C^m,\Omega',g} < \varepsilon &\Longrightarrow \|g-g'\|_{C^m,\Omega',\mathcal{U},\mathcal{K}} < C\varepsilon \leq \varepsilon' \\ &\Longrightarrow \|g^{(m)}-g'^{(m)}\|_{C^0,\Omega^{(m)},\mathcal{U}^{(m)},\mathcal{K}^{(m)}} < \delta/C^{(m)} \\ &\Longrightarrow \delta_0 := \|g^{(m)}-g'^{(m)}\|_{C^0,\Omega^{(m)},g^{(m)}} < \delta\ .\end{aligned}$$

For any $\lambda \in [1, e^r)$ such that $\delta_0 \leq \min\{1-\lambda^{-2}, \lambda^2-1\}$, we have $\frac{1}{\lambda}|\xi|^{(m)} \leq |\xi|'^{(m)} \leq \lambda|\xi|^{(m)}$ for all $\xi \in T\Omega^{(m)}$ by Lemma 5.4.1-(ii), where $|\ |^{(m)}$ and $|\ |'^{(m)}$ denote the norms defined by $g^{(m)}$ and $g'^{(m)}$, respectively. So ϕ is an (m, R, λ)-pointed local quasi-isometry $(M, x) \rightarrowtail (N, y)$, and therefore $[N, y] \in U^{(m)}_{R,r}(M, x)$. □

Proposition 5.4.5. (i) *For all $R, r > 0$, if $e^{2\varepsilon} - e^{-2\varepsilon} \leq r$, then $U^0_{R,\varepsilon} \subset D^0_{R,r}$.*

(ii) *For all $m \in \mathbb{Z}^+$, $R, r > 0$ and $[M, x] \in \mathcal{M}_*(n)$, there is some $\varepsilon > 0$ such that $U^m_{R,\varepsilon}(M, x) \subset D^m_{R,r}(M, x)$.*

Proof. Let us show (i). If $([M, x], [N, y]) \in U^0_{R,\varepsilon}$, then there is a $(0, R, \lambda)$-pointed local quasi-isometry $\phi : (M, x) \rightarrowtail (N, y)$ for some $\lambda \in [1, e^\varepsilon)$. Set $g = g_M$ and $g' = \phi^* g_N$, and let $|\ |$ and $|\ |'$ denote the respective norms. Thus there is some compact domain $\Omega \subset \operatorname{dom}\phi$ such that $B_M(x, R) \subset \Omega$ and $\frac{1}{\lambda}|\xi| \leq |\xi|' \leq \lambda|\xi|$ for all $\xi \in T\Omega$. By Lemma 5.4.1-(i), it follows that

$$\|g-g'\|_{C^0,\Omega,g} \leq \lambda^2 - \lambda^{-2} < e^{2\varepsilon} - e^{-2\varepsilon} \leq r\ .$$

So $([M, x], [N, y]) \in D^0_{R,r}$.

Let us prove (ii). Let $m \in \mathbb{Z}^+$, $R, r > 0$ and $[M, x] \in \mathcal{M}_*(n)$. Take $\mathcal{U}$, $\mathcal{K}$, K, $\mathcal{U}^{(m)}$, $\mathcal{K}^{(m)}$ and $K^{(m)}$ like in the proof of Proposition 5.4.4-(ii). Let $\varepsilon > 0$, to be fixed later. For any $[N, y] \in U^m_{R,\varepsilon}(M, x)$, there is an (m, R, λ)-pointed local quasi-isometry $\phi : (M, x) \rightarrowtail (N, y)$ for some $\lambda \in [1, e^\varepsilon)$. Again, let $g = g_M$ and $g' = \phi^* g_N$. Thus there is a compact domain $\Omega^{(m)} \subset \operatorname{dom}\phi^{(m)}_* \cap \operatorname{Int}(K^{(m)})$ so that $B^{(m)}_M(x, R) \subset \Omega^{(m)}$ and $\frac{1}{\lambda}|\xi|^{(m)} \leq |\xi|'^{(m)} \leq \lambda|\xi|^{(m)}$ for all $\xi \in T\Omega^{(m)}$, where $|\ |^{(m)}$ and $|\ |'^{(m)}$ denote the norms defined by $g^{(m)}$ and $g'^{(m)}$, respectively. By continuity, given any $\lambda' \in (\lambda, e^\varepsilon)$, there is some compact domain $\Omega'^{(m)} \subset \operatorname{dom}\phi^{(m)}_* \cap K^{(m)}$ such that $\Omega^{(m)} \subset \operatorname{Int}(\Omega'^{(m)})$ and $\frac{1}{\lambda'}|\xi|^{(m)} \leq |\xi|'^{(m)} \leq \lambda'|\xi|^{(m)}$ for all $\xi \in \Omega'^{(m)}$. By Lemma 5.4.1-(i), it follows that

$$\|g^{(m)}-g'^{(m)}\|_{C^0,\Omega'^{(m)},g^{(m)}} \leq \lambda'^2 - \lambda'^{-2} < e^{2\varepsilon} - e^{-2\varepsilon}\ .$$

There is some compact domain $\Omega \subset M$ such that $\Omega^{(m)} \cap M \subset \Omega \subset \operatorname{Int}(\Omega'^{(m)})$. Thus $\Omega \subset \Omega'^{(m)} \cap M \subset K^{(m)} \cap M = K$, and

$$B_M(x, R) = B^{(m)}_M(x, R) \cap M \subset \Omega^{(m)} \cap M \subset \Omega$$

by Remark 4.2.1-(ii). Take some $C \geq 1$ satisfying (4.3) with $\mathcal{U}$, $\mathcal{K}$, Ω and g, and some $C^{(m)} \geq 1$ satisfying (4.3) with $\mathcal{U}^{(m)}$, $\mathcal{K}^{(m)}$, $\Omega'^{(m)}$ and $g^{(m)}$. For $\rho > 0$ and $n+1 \leq \mu \leq 2^m n$, let $\sigma^{(m)}_{a,\rho,\mu} : U_a \to U^{(m)}_a$ be the section of each projection $\pi : U^{(m)}_a \to U_a$ of the type used in Lemma 4.2.1-(ii). Since $\Omega \subset \operatorname{Int}(\Omega'^{(m)})$, there is some $\rho > 0$ so that $\sigma^{(m)}_{\rho,\mu}(K_a \cap \Omega) \subset K^{(m)}_a \cap \Omega'^{(m)}$ for all a and μ. Thus, by Lemma 4.2.1-(ii), there is some $\varepsilon' > 0$, depending on r and ρ, such that

$$\|g^{(m)} - g'^{(m)}\|_{C^0,\Omega'^{(m)},\mathcal{U}^{(m)},\mathcal{K}^{(m)}} < \varepsilon' \implies \|g - g'\|_{C^m,\Omega,\mathcal{U},\mathcal{K}} < r/C \ .$$

Suppose that $e^{2\varepsilon} - e^{-2\varepsilon} \leq \varepsilon'/C^{(m)}$. Then

$$\begin{aligned}\|g^{(m)} - g'^{(m)}\|_{C^0,\Omega'^{(m)},g^{(m)}} &< e^{2\varepsilon} - e^{-2\varepsilon}\\ &\implies \|g^{(m)} - g'^{(m)}\|_{C^0,\Omega'^{(m)},\mathcal{U}^{(m)},\mathcal{K}^{(m)}} < C^{(m)}(e^{2\varepsilon} - e^{-2\varepsilon}) \leq \varepsilon'\\ &\implies \|g - g'\|_{C^m,\Omega,\mathcal{U},\mathcal{K}} < r/C \implies \|g - g'\|_{C^m,\Omega,g} < r \ ,\end{aligned}$$

showing that $[N, y] \in D^{(m)}_{R,r}(M, x)$. □

Corollary 5.4.6. *The C^∞ convergence in $\mathcal{M}_*(n)$ describes the C^∞ topology.*

Proof. This is a direct consequence of Remark 5.4.1 and Propositions 5.4.4 and 5.4.5. □

5.5 $\mathcal{M}^\infty_*(n)$ is Polish

Proposition 5.5.1. *$\mathcal{M}^\infty_*(n)$ is separable.*

Proof. The isometry classes of pointed compact Riemannian manifolds form a subspace, $\mathcal{M}^\infty_{*,\mathrm{c}}(n) \subset \mathcal{M}^\infty_*(n)$, which is dense because, for all $[M, x] \in \mathcal{M}^\infty_*(n)$ and $R > 0$, the ball $B_M(x, R)$ can be isometrically embedded in a compact Riemannian manifold.

As a consequence of the finiteness theorems of Cheeger on Riemannian manifolds [25], it follows that there are countably many diffeomorphism classes of compact C^∞ manifolds (see [60, Corollary 37, p. 320] or [24, Theorem IX.8.1]). Thus there is a countable family $\mathcal{C}$ of C^∞ compact manifolds containing exactly one representative of every diffeomorphism class.

For every $M \in \mathcal{C}$, the set of metrics on M, $\operatorname{Met}(M)$, is an open subspace of the space of smooth sections, $C^\infty(M; T^*M \odot T^*M)$, with the C^∞ topology, where "$\odot$" denotes the symmetric product. Then, since $C^\infty(M; T^*M \odot T^*M)$ is separable, we can choose a countable dense subset $\mathcal{G}_M \subset \operatorname{Met}(M)$. Choose also a countable dense subset $\mathcal{D}_M \subset M$.

Clearly, the countable set

$$\{\, [(M, g), x] \mid M \in \mathcal{C},\ g \in \mathcal{G}_M,\ x \in \mathcal{D}_M \,\}$$

is dense in $\mathcal{M}^\infty_{*,\mathrm{c}}(n)$, and therefore it is also dense in $\mathcal{M}^\infty_*(n)$. □

Remark 5.5.1. Observe that the proof of Proposition 5.5.1 shows that $\mathcal{M}_{*,\mathrm{c}}^\infty(n)$ is dense in $\mathcal{M}_*^\infty(n)$.

Proposition 5.5.2. $\mathcal{M}_*^\infty(n)$ *is completely metrizable.*

Proof. The C^∞ uniformity on $\mathcal{M}_*(n)$ is metrizable because it is separating and has a countable base of entourages [77, Corollary 38.4]. Thus it is enough to check that the C^∞ uniformity on $\mathcal{M}_*(n)$ is complete.

Consider an arbitrary Cauchy sequence $[M_i, x_i]$ in $\mathcal{M}_*(n)$ with respect to the C^∞ uniformity. We have to prove that $[M_i, x_i]$ is convergent in $\mathcal{M}_*^\infty(n)$. By taking a subsequence if necessary, we can suppose that $([M_i, x_i], [M_{i+1}, x_{i+1}]) \in U_{R_i,r_i}^{m_i}$ for sequences, $m_i \uparrow \infty$ in $\mathbb{N}$, and $R_i \uparrow \infty$ and $r_i \downarrow 0$ in $\mathbb{R}^+$, such that $\sum_i r_i < \infty$, and $R_{i+1} \geq e^{r_i} R_i$ for all i. Let $\bar{r}_i = \sum_{j \geq i} r_j$. Consider other sequences $R_i', R_i'' \uparrow \infty$ in $\mathbb{R}^+$ such that $R_i' < R_i'' \leq e^{-\bar{r}_i} R_i$ and $R_{i+1}' \geq e^{r_i} R_i''$.

For each i, there is some $\lambda_i \in (1, e^{r_i})$ and some (m_i, R_i, λ_i)-pointed local quasi-isometry $\phi_i \colon (M_i, x_i) \rightarrowtail (M_{i+1}, x_{i+1})$, which can be assumed to be C^∞ by Remark 5.2.1-(iii). Then $\bar{\lambda}_i := \prod_{j \geq i} \lambda_j < e^{\bar{r}_i} < \infty$. For $i < j$, the pointed local quasi-isometry $\psi_{ij} = \phi_{j-1} \cdots \phi_i : (M_i, x_i) \rightarrowtail (M_j, x_j)$ is of type $(m_i, R_i/\bar{\lambda}_i, \bar{\lambda}_i)$ by Lemma 5.2.3-(i).

For $i, m \in \mathbb{N}$, let

$$
\begin{aligned}
B_i &= B_i(x_i, R_i)\ , & B_i' &= B_i(x_i, R_i')\ , & B_i'' &= B_i(x_i, R_i'')\ ,\\
B_i^{(m)} &= B_i^{(m)}(x_i, R_i)\ , & B_i'^{(m)} &= B_i^{(m)}(x_i, R_i')\ , & B_i''^{(m)} &= B_i^{(m_i)}(x_i, R_i'')\ .
\end{aligned}
$$

A bar will be added to this notation when the corresponding closed balls are considered. We have $\phi_i(\overline{B}_i) \subset B_{i+1}$ because $R_{i+1} > \lambda_i R_i$, and $\phi_{i*}^{(m_i)}(\overline{B}_i''^{(m_i)}) \subset B_{i+1}'^{(m_i)} \subset B_{i+1}'^{(m_{i+1})}$ since $R_{i+1}' > \lambda_i R_i''$ and by Remark 4.2.1-(i). Furthermore $B_i'' \subset \operatorname{dom} \psi_{ij}$ and $B_i''^{(m_i)} \subset \operatorname{dom} \psi_{ij*}^{(m_i)}$ for $i < j$ because $R'' \leq R_i/\bar{\lambda}_i$. Therefore $\psi_{ij}(B_i) \subset B_j$ and $\psi_{ij*}^{(m_i)}(B_i''^{(m_i)}) \subset B_j'^{(m_j)}$.

The restrictions $\psi_{ij} : B_i \to B_j$ form a direct system of spaces, whose direct limit is denoted by $\widehat{M}$. Let $\psi_i : B_i \to \widehat{M}$ be the induced maps, whose images, $\widehat{B}_i := \psi_i(B_i)$, form an exhausting increasing sequence of subsets of $\widehat{M}$. All points $\psi_i(x_i)$ are equal in $\widehat{M}$, and will be denoted by $\hat{x}$. The space $\widehat{M}$ is connected because it is the union of the connected subspaces $\widehat{B}_i$ whose intersection contains $\hat{x}$. By the definition of the direct limit and since the maps ψ_{ij} are open embeddings, it follows that all maps ψ_i are open embeddings, and therefore $\widehat{M}$ is a Hausdorff n-manifold. Equip each $\widehat{B}_i$ with the C^∞ structure that corresponds to the C^∞ structure of B_i by ψ_i. These C^∞ structures are compatible one another because the open embeddings ψ_{ij} are C^∞, and therefore they define a C^∞ structure on $\widehat{M}$. Moreover let $\hat{g}_i$ be the Riemannian metric on each $\widehat{B}_i$ that corresponds to $g_i|_{B_i}$ via ψ_i.

Take some compact domains, Ω_i in every M_i and $\Omega_i^{(m_i)}$ in $T^{(m_i)}M_i$, such that $B'_i \subset \Omega_i \subset \operatorname{Int}(\Omega_i^{(m_i)})$ and $B_i'^{(m_i)} \subset \Omega_i^{(m_i)} \subset B_i''^{(m_i)}$; thus $\Omega_i \subset B''_i$ by Remark 4.2.1-(ii). Let $\widehat{\Omega}_i = \psi_i(\Omega_i)$.

Claim 5.5.1. $\widehat{M} = \bigcup_i \widehat{\Omega}_i$.

This equality holds because, for each i, there is some j so that $R'_j > \bar{\lambda}_i R_i$, obtaining

$$\psi_{ij}(B_i) \subset B_j(x_j, \bar{\lambda}_i R_i) \subset B'_j \subset \Omega_j \ ,$$

and therefore $\widehat{B}_i = \psi_j\psi_{ij}(B_i) \subset \psi_j(\Omega_j) = \widehat{\Omega}_j$.

Claim 5.5.2. For all i, the restrictions $\hat{g}_j|_{\widehat{\Omega}_i}$, with $j \geq i$, form a convergent sequence in the space of C^{m_i} sections, $C^{m_i}(\widehat{\Omega}_i; T\widehat{\Omega}_i^* \odot T\widehat{\Omega}_i^*)$, with the C^{m_i} topology, and its limit, $\hat{g}_{i,\infty}$, is positive definite at every point.

Clearly, Claim 5.5.2 follows by showing that the restrictions of the metrics $g_{ij} := \psi_{ij}^* g_j$ to Ω_i, for $j \geq i$, form a convergent sequence in $C^{m_i}(\Omega_i; T\Omega_i^* \odot T\Omega_i^*)$, and its limit, $g_{i,\infty}$, is positive definite at every point. To begin with, let us show that $g_{ij}|_{\Omega_i}$ is a Cauchy sequence with respect to $\| \ \|_{C^{m_i},\Omega_i,g_i}$.

We have

$$\frac{1}{\bar{\lambda}_i} |\xi|_i^{(m_i)} \leq |\xi|_{ij}^{(m_i)} \leq \bar{\lambda}_i \, |\xi|_i^{(m_i)} \tag{5.6}$$

for all $\xi \in T\Omega_i^{(m_i)}$, where $| \ |_i^{(m_i)}$ and $| \ |_{ij}^{(m_i)}$ are the norms defined by $g_i^{(m_i)}$ and $g_{ij}^{(m_i)}$, respectively. By Lemma 5.4.1-(i), it follows that

$$\|g_i^{(m_i)} - g_{ij}^{(m_i)}\|_{C^0,\Omega_i^{(m_i)},g_i^{(m_i)}} \leq \bar{\lambda}_i^2 - \bar{\lambda}_i^{-2} \ .$$

Then, for $k \geq j$,

$$\begin{aligned}
\|g_{ij}^{(m_i)} - g_{ik}^{(m_i)}\|_{C^0,\Omega_i^{(m_i)},g_{ij}^{(m_i)}} &= \|g_j^{(m_i)} - g_{jk}^{(m_i)}\|_{C^0,\psi_{ij*}^{(m_i)}(\Omega_i^{(m_i)}),g_j^{(m_i)}} \\
&\leq \|g_j^{(m_j)} - g_{jk}^{(m_j)}\|_{C^0,\Omega_j^{(m_j)},g_j^{(m_j)}} \leq \bar{\lambda}_j^2 - \bar{\lambda}_j^{-2}
\end{aligned} \tag{5.7}$$

because

$$\psi_{ij*}^{(m_i)}(\Omega_i^{(m_i)}) \subset \psi_{ij*}^{(m_i)}(B_i''^{(m_i)}) \subset B_j'^{(m_j)} \subset \Omega_j^{(m_j)}$$

and $g_{jk}^{(m_j)} = g_{jk}^{(m_i)}$ on $\Omega_j^{(m_j)} \cap B_j^{(m_i)} \supset \psi_{ij*}^{(m_i)}(\Omega_i^{(m_i)})$ (Remark 4.2.1-(i)). We get

$$\|g_{ij}^{(m_i)} - g_{ik}^{(m_i)}\|_{C^0,\Omega_i^{(m_i)},g_i^{(m_i)}} \leq \bar{\lambda}_i^2(\bar{\lambda}_j^2 - \bar{\lambda}_j^{-2}) \tag{5.8}$$

by (5.6), (5.7) and Lemma 5.4.1-(iii).

Let $\mathcal{U}_i$ be a finite collection of charts of M_i with domains $U_{i,a}$, and let $\mathcal{K}_i = \{K_{i,a}\}$ be a family of compact subsets of M_i, with the same index set as $\mathcal{U}_i$, such that $K_{i,a} \subset U_{i,a}$ for all a, and $\overline{B}''_i \subset \bigcup_a K_{i,a} =: K_i$. Thus $\Omega_i \subset K_i$. With the notation of Section 4.2, let

$\mathcal{U}_i^{(m_i)}$ be the family of induced charts of $T^{(m_i)}M_i$ with domains $U_{i,a}^{(m_i)}$. Like in the proof of Proposition 5.4.4-(ii), let $\mathcal{K}_i^{(m_i)}$ be the family of compact subsets

$$K_{i,a}^{(m_i)} = \{\, \xi \in B_i^{(m_i)} \mid \pi(\xi) \in K_{i,a},\ d_i^{(m_i)}(\xi, \pi_i(\xi)) \leq R_i''' \,\} \subset U_{i,a}^{(m_i)}\ ,$$

for some $R_i''' > R_i''$, where $\pi : B_i^{(m_i)} \to B_i$. We have $B_i^{\prime\prime(m_i)} \subset \bigcup_a K_{i,a}^{(m_i)} =: K_i^{(m_i)}$. Hence $\Omega_i^{(m_i)} \subset K_i^{(m_i)}$.

Choose some $C_i \geq 1$ satisfying (4.3) with $\mathcal{U}_i$, $\mathcal{K}_i$, Ω_i and g_i, and some $C_i^{(m_i)} \geq 1$ satisfying (4.3) with $\mathcal{U}_i^{(m_i)}$, $\mathcal{K}_i^{(m_i)}$, $\Omega_i^{(m_i)}$ and $g^{(m_i)}$. For any $\rho > 0$ and $n+1 \leq \mu \leq 2^{m_i}n$, let $\sigma_{i,a,\rho,\mu}^{(m_i)} : U_{i,a} \to U_{i,a}^{(m_i)}$ be the section of each projection $\pi : U_{i,a}^{(m_i)} \to U_{i,a}$ of the type used in Lemma 4.2.1-(ii). Since $\Omega_i \subset \operatorname{Int}(\Omega_i^{(m_i)})$, there is some $\rho > 0$ so that $\sigma_{i,a,\rho,\mu}^{(m_i)}(K_{i,a} \cap \Omega_i) \subset K_{i,a}^{(m_i)} \cap \Omega_i^{(m_i)}$ for all a and μ. Thus, by Lemma 4.2.1-(ii), given any $\varepsilon > 0$, there is some $\delta > 0$, depending on ε and ρ, such that

$$\|g_{ij}^{(m_i)} - g_{ik}^{(m_i)}\|_{C^0,\Omega_i^{(m_i)},\mathcal{U}_i^{(m_i)},\mathcal{K}_i^{(m_i)}} < \delta \Longrightarrow \|g_{ij} - g_{ik}\|_{C^{m_i},\Omega_i,\mathcal{U}_i,\mathcal{K}_i} < \varepsilon / C_i\ . \tag{5.9}$$

Since $\bar\lambda_j \downarrow 1$, we have $\bar\lambda_i^2(\bar\lambda_j^2 - \bar\lambda_j^{-2}) < \delta / C_i^{(m_i)}$ for j large enough, giving

$$\begin{aligned} \|g_{ij}^{(m_i)} - g_{ik}^{(m_i)}\|_{C^0,\Omega_i^{(m_i)},g_i^{(m_i)}} < \delta / C_i^{(m_i)} &\Longrightarrow \|g_{ij}^{(m_i)} - g_{ik}^{(m_i)}\|_{C^0,\Omega_i^{(m_i)},\mathcal{U}_i^{(m_i)},\mathcal{K}_i^{(m_i)}} < \delta \\ &\Longrightarrow \|g_{ij} - g_{ik}\|_{C^{m_i},\Omega_i,\mathcal{U}_i,\mathcal{K}_i} < \varepsilon / C_i \Longrightarrow \|g_{ij} - g_{ik}\|_{C^{m_i},\Omega_i,g_i} < \varepsilon \end{aligned}$$

by (5.8), (5.9) and (4.3). This shows that $g_{ij}|_{\Omega_i}$ is a Cauchy sequence in the Banach space $C^{m_i}(\Omega_i; T\Omega_i^* \odot T\Omega_i^*)$ with $\|\ \|_{C^{m_i},\Omega_i,g_i}$, and therefore it has a limit $g_{i,\infty}$. For all nonzero $\xi \in T\Omega_i$, we have

$$g_{i,\infty}(\xi,\xi) = \lim_j g_{ij}(\xi,\xi) \geq \frac{1}{\bar\lambda_i}\, g_i(\xi,\xi) > 0\ ,$$

obtaining that $g_{i,\infty}$ is positive definite. This completes the proof of Claim 5.5.2.

According to Claim 5.5.2, each $\hat g_{i,\infty}$ is a C^{m_i} Riemannian metric on $\widehat\Omega_i$, and, obviously, $\hat g_{j,\infty}|_{\widehat\Omega_i} = \hat g_{i,\infty}$ for $j > i$. Hence the metric tensors $\hat g_{i,\infty}$ can be combined to define a C^∞ Riemannian metric $\hat g$ on $\widehat M$ by Claim 5.5.1.

Let $|\ |_{i,\infty}^{(m_i)}$ be the norm defined by $g_{i,\infty}^{(m_i)}$ on $T\Omega_i^{(m_i)}$. By (5.6) and because $|\ |_{i,\infty}^{(m_i)} = \lim_j |\ |_{ij}^{(m_i)}$ on $T\Omega_i^{(m_i)}$, we get $\frac{1}{\bar\lambda_i}\,|\xi|_i^{(m_i)} \leq |\xi|_{i,\infty}^{(m_i)} \leq \bar\lambda_i\,|\xi|_i^{(m_i)}$ for all $\xi \in T\Omega_i^{(m_i)}$. Thus, by Remark 5.1.1-(iii), Ω_i contains the $g_{i,\infty}$-ball of center x_i and radius $R_i'/\bar\lambda_i$ because it contains B_i'; in particular, $\widehat M$ is complete because $R_i'/\bar\lambda_i \to \infty$ and every Ω_i is compact. Since $g_{i,\infty} = \psi_i^*\hat g$, it also follows that $\psi_{i*}^{(m_i)} : \Omega_i^{(m_i)} \to T^{(m_i)}\widehat M$ is a $\bar\lambda_i$-quasi-isometry. So $\psi_i : (M_i, x_i) \rightarrowtail (\widehat M, \hat x)$ is an $(m_i, R_i', \bar\lambda_i)$-pointed local quasi-isometry, obtaining that $([M_i,x_i],[\widehat M,\hat x]) \in U_{R_i',s_i}^{m_i}$ for any sequence $s_i \downarrow 0$ with $\bar\lambda_i < e^{s_i}$, and therefore $[M_i,x_i] \to [\widehat M,\hat x]$ as $i \to \infty$ in $\mathcal{M}_*^\infty(n)$. □

Corollary 5.5.3. $\mathcal{M}_*^\infty(n)$ *is Polish.*

Proof. This is the content of Propositions 5.5.1 and 5.5.2 together. □

Corollaries 5.4.6 and 5.5.3 give Theorem 1.3.2.

5.6 Some basic properties of $\mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$

For each closed C^∞ manifold M of dimension ≥ 2, the non-periodic metrics on M form a residual subset of $\mathrm{Met}(M)$ with the C^∞ topology [12, Corollary 3.5], [75, Proposition 1]. Then, since $\mathcal{M}^\infty_{*,\mathrm{c}}(n)$ is dense in $\mathcal{M}^\infty_*(n)$ (Remark 5.5.1), it follows that $\mathcal{M}^\infty_{*,\mathrm{np}}(n)$ is dense in $\mathcal{M}^\infty_*(n)$, and therefore $\mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$ is dense in $\mathcal{M}^\infty_*(n)$ too. On the other hand, $\mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$ is G_δ in $\mathcal{M}^\infty_*(n)$ by Lemmas 5.6.1 and 5.6.3 below, and therefore it is a Polish subspace [51, Theorem I.3.11]. This proves Theorem 1.3.3-(i).

Lemma 5.6.1. *For every $n \in \mathbb{Z}^+$ and $[M,x] \in \mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$, there is some $r > 0$ such that, if*

$$\{\, h \in \mathrm{Iso}(M) \mid h(x) \in \overline{B}(x,r) \,\} = \{\mathrm{id}_M\}\ ,$$

then there is some neighborhood $\mathcal{L}$ of $[M,x]$ in $\mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$ so that

$$\{\, h \in \mathrm{Iso}(L) \mid h(y) \in \overline{B}(y,r) \,\} = \{\mathrm{id}_L\}$$

for all $[L,y] \in \mathcal{L}$.

Proof. Suppose that the statement is false. Then there is some convergent sequence, $[M_i,x_i] \to [M,x]$, in $\mathcal{M}^\infty_*(n)$ so that, for each i, some $h_i \in \mathrm{Iso}(M_i) \smallsetminus \{\mathrm{id}_{M_i}\}$ satisfies $h_i(x_i) \in \overline{B}_i(x_i,r)$. Choose any sequence of compact domains Ω_q of M such that $\overline{B}(x,2r) \subset \mathrm{Int}(\Omega_q)$ and $d(x,\partial\Omega_q) \to \infty$ as $q \to \infty$. For each q and i large enough, there is some pointed smooth embedding $\phi_{q,i} : (\Omega_q,x) \to (M_i,x_i)$ so that $\phi^*_{q,i}g_i \to g|_{\Omega_q}$ as $i \to \infty$ with respect to the C^∞ topology. Thus $\overline{B}_i(x_i,2r) \subset \phi_{q,i}(\mathrm{Int}(\Omega_q))$ for i large enough.

Claim 5.6.1. If r is small enough, we can assume that there is some $\delta > 0$ such that, for i large enough, the maps h_i can be chosen so that $d_i(z_i,h_i(z_i)) \geq \delta$ for some $z_i \in B_i(x_i,r)$.

Given any index i, suppose first that there is some $k \in \mathbb{Z} \smallsetminus \{0\}$ such that $h_i^k(x_i) \notin \overline{B}_i(x_i,r/2)$. Then there is some $k \in \mathbb{Z} \smallsetminus \{0\}$ such that $h_i^k(x_i) \notin \overline{B}_i(x_i,r/2)$ and $h_i^\ell(x_i) \in \overline{B}_i(x_i,r/2)$ if $|\ell| < |k|$. If $k = 1$, then $d_i(x_i,h_i(x_i)) \geq r/2$. If $k = -1$, then

$$d_i(x_i,h_i(x_i)) = d_i\left(h_i^{-1}(x_i),x_i\right) \geq r/2$$

as well. If $|k| \geq 2$, then there is some $\ell \in \mathbb{Z}$ such that $|\ell|, |k-\ell| < |k|$. Hence

$$d_i\left(x_i,h_i^k(x_i)\right) \leq d_i\left(x_i,h_i^\ell(x_i)\right) + d_i\left(h_i^\ell(x_i),h_i^k(x_i)\right) = d_i\left(x_i,h_i^\ell(x_i)\right) + d_i\left(x_i,h_i^{k-\ell}(x_i)\right) \leq r\ .$$

Therefore, by using h_i^k instead of h_i, we can assume that $d_i(x_i,h_i(x_i)) \geq r/2$ in this case.

Now, suppose that $h_i^k(x_i) \in \overline{B}_i(x_i,r/2)$ for all $k \in \mathbb{Z}$. Consider the non-trivial abelian subgroup $A_i = \overline{\{\, h_i^k \mid k \in \mathbb{Z} \,\}} \subset \mathrm{Iso}(M_i)$. Since $a(x_i) \in \overline{B}_i(x_i,r/2)$ for any $a \in A_i$, it follows that A_i is compact in the C^∞ topology by Proposition 5.1.11, and thus A_i is a non-trivial compact abelian Lie subgroup of $\mathrm{Iso}(M_i)$. Let μ_i be a bi-invariant probability measure on A_i, and let $f_i : A_i \to M$ be the mass distribution defined by $f_i(a) = a(x_i)$. By the C^∞ convergence $\phi_{q,i}^* g_i \to g|_{\Omega_q}$, we can suppose that r is so small that the ball $B_i(x_i, 2r/3)$ of M_i satisfies the conditions of Proposition 5.8.2 for i large enough. Then, since $f_i(A_i) \subset \overline{B}_i(x_i, r/2) \subset B_i(x_i, 2r/3)$, the center of mass $y_i = \mathcal{C}_{f_i}$ is defined in $B_i(x_i, 2r/3)$. Moreover y_i is a fixed point of the canonical action of A_i on M [50, Section 2.1]. Since there is a neighborhood of the identity in the orthogonal group $\mathrm{O}(n)$ which contains no non-trivial subgroup (simply because $\mathrm{O}(n)$ is a Lie group), it follows that there is some $K > 0$ such that, for any non-trivial subgroup $A \subset \mathrm{O}(n)$, there is some $a \in A$ and some $v \in \mathbb{R}^n$ such that $|v| = 1$ and $|a(v) - v| \geq K$. In our setting, the subgroup $\{ a_{*y_i} \mid a \in A_i \}$ of the orthogonal group $\mathrm{O}(T_{y_i} M_i) \equiv \mathrm{O}(n)$ is non-trivial because M_i is connected and A_i is non-trivial. Hence there is some $a_i \in A_i$ and some $\xi_i \in T_{y_i} M_i$ such that $|\xi_i| = 1$ and $|a_{i*}(\xi_i) - \xi_i| \geq K$. By the C^∞ convergence $\phi_{q,i}^* g_i \to g|_{\Omega_q}$, we can also assume that r is so small that there exists some $C \geq 1$ such that $\exp_{y_i} : B(0_{y_i}, r) \to B(y_i, r)$ is C-quasi-isometric for i large enough. Then, for $z_i = \exp_{y_i}(\frac{r}{3}\,\xi_i) \in \overline{B}_i(y_i, r/3) \subset B_i(x_i, r)$, we get

$$d_i(z_i, a_i(z_i)) \geq \frac{r}{3C}\,|\xi_i - h'_{i*}(\xi_i)| \geq \frac{rK}{3C}\,.$$

Thus, by using a_i instead of h_i, we can assume in this case that $d_i(z_i, h_i(z_i)) \geq rK/3C$. Therefore Claim 5.6.1 follows with $\delta = \min\{r/2, rK/3C\}$.

For each q, we can assume that

$$\overline{B}(x, \operatorname{diam}(\Omega_q) + r) \subset \operatorname{Int}(\Omega_{q+1})\,,$$

obtaining

$$\overline{B}_i(x_i, \operatorname{diam}(\phi_{q,i}(\Omega_q)) + r) \subset \operatorname{Int}(\phi_{q+1,i}(\Omega_{q+1}))$$

for all i large enough by the C^∞ convergence $\phi_{q,i}^* g_i \to g|_{\Omega_q}$. Then $h'_{q,i} := \phi_{q+1,i}^{-1}\, h_i\, \phi_{q,i} : \Omega_q \to M$ is well defined for each q and all i large enough because $x_i \in \phi_{q,i}(\Omega_q)$ and $h_i(x_i) \in \overline{B}_i(x_i, r)$. On the one hand, from the C^∞ convergence $\phi_{q,i}^* g_i \to g|_{\Omega_q}$ and since $h_i(x_i) \in \overline{B}_i(x_i, r)$, we get the C^∞ convergence $h'^*_{q,i} g \to g|_{\Omega_q}$ and $\limsup_i d(x, h'_{q,i}(x)) \leq r$; in particular, for each q, the maps $h'_{q,i}$ are equi-quasi-isometries of order ∞. Therefore, by Proposition 5.1.11, some subsequence of $h'_{q,i}$ is C^∞ convergent to some C^∞ map $h'_q : \Omega_q \to M$, which is an isometric embedding satisfying $h'_q(x) \in \overline{B}(x, r)$.

For all $p \geq q$, the restrictions $h'_p|_{\Omega_q}$ form a sequence of isometric embeddings satisfying $h'_p(x) \in \overline{B}(x, r)$. Then, by Proposition 5.1.11, there is some sequence of positive integers $p(q, k)$ for each q so that the subsequence $h'_{p(q,k)}|_{\Omega_q}$ of $h'_p|_{\Omega_q}$ is C^∞ convergent as $k \to \infty$ to an isometric embedding $h''_q : \Omega_q \to M$ satisfying $h''_q(x) \in \overline{B}(x, r)$. We can assume that

$p(q+1,k)$ is a subsequence of $p(q,k)$ for each q, yielding $h''_{q+1}|_{\Omega_q} = h''_q$. So the maps h''_q can be combined to define an isometry $h : M \to M$ satisfying $h(x) \in \overline{B}(x,r)$.

Now, fix any q and let $z'_{p,i} = \phi_{p,i}^{-1}(z_i)$ for each $p \geq q$ and all i large enough. From $z_i \in B_i(x_i,r)$ and the C^∞ convergence $\phi_{p,i}^* g_i \to g|_{\Omega_p}$, it follows that $z'_{p,i}$ approaches the compact set $\overline{B}(x,r)$ as $i \to \infty$. Then, for each $p \geq q$, there is a sequence $z_{p,i}$ in $B(x,r)$ so that $d(z_{p,i}, z'_{p,i}) \to 0$. Hence, by the C^∞ convergence $\phi_{p,i}^* g_i \to g|_{\Omega_p}$ and Claim 5.6.1, we get

$$
\begin{aligned}
\sup\{\, d(z,h(z)) \mid z \in B(x,r) \,\} &= \sup\{\, d(z,h''_q(z)) \mid z \in B(x,r) \,\} \\
&\geq \sup\left\{ \liminf_p d(z,h'_p(z)) \mid z \in B(x,r) \right\} \\
&\geq \sup\left\{ \liminf_p \liminf_i d(z,h'_{p,i}(z)) \mid z \in B(x,r) \right\} \\
&\geq \liminf_p \liminf_i d(z_{p,i}, h'_{p,i}(z_{p,i})) \\
&= \liminf_p \liminf_i d(z'_{p,i}, h'_{p,i}(z'_{p,i})) \\
&\geq \liminf_i d_i(z_i, h_i(z_i)) \geq \delta \;.
\end{aligned}
$$

So $h \neq \mathrm{id}_M$, which is a contradiction because $h(x) \in \overline{B}(x,r)$. □

Lemma 5.6.2. *For $n \geq 2$ and each point $[M,x] \in \mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$, there is some $r > 0$ such that, for each $\varepsilon \in (0,r)$, there is some neighborhood $\mathcal{N}$ of $[M,x]$ in $\mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$ so that, if an equivalence class $\iota(L)$ of $\mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$ meets $\mathcal{N}$ at points $[L,y]$ and $[L,z]$, then either $d_L(y,z) < \varepsilon$ or $d_L(y,z) > r$.*

Proof. Since M is locally non-periodic, there is some $r > 0$ such that

$$\{\, h \in \mathrm{Iso}(M) \mid d(x,h(x)) \leq r \,\} = \{\mathrm{id}_M\} \;. \tag{5.10}$$

Suppose that the statement is false for this r. Then, given any $\varepsilon \in (0,r)$, there are sequences $[L_i,y_i]$ and $[L_i,z_i]$ in $\mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$ converging to $[M,x]$ in $\mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$ such that $\varepsilon \leq d_i(y_i,z_i) \leq r$ for all i.

Take a sequence of compact domains Ω_q of M such that $x \in \Omega_q$ and $d(x,\partial\Omega_q) \to \infty$ as $q \to \infty$. For each q, there are C^∞ embeddings $\phi_{q,i} : \Omega_q \to M_i$ and $\psi_{q,i} : \Omega_q \to M_i$ for i large enough so that $\phi_{q,i}(x) = y_i$, $\psi_{q,i}(x) = z_i$, and $\phi_{q,i}^* g_i, \psi_{q,i}^* g_i \to g|_{\Omega_q}$ as $i \to \infty$ with respect to the C^∞ topology. We can also assume that, for each q,

$$\overline{B}(x, \mathrm{diam}(\Omega_q) + r) \subset \mathrm{Int}(\Omega_{q+1}) \;,$$

giving

$$\phi_{q,i}(\Omega_q) \subset \overline{B}_i(y_i, \mathrm{diam}(\phi_{q,i}(\Omega_q))) \subset \overline{B}_i(z_i, \mathrm{diam}(\phi_{q,i}(\Omega_q)) + r) \subset \mathrm{Int}(\psi_{q+1,i}(\Omega_{q+1}))$$

for i large enough by the C^∞ convergence $\phi_{q,i}^*g_i, \psi_{q,i}^*g_i \to g|_{\Omega_q}$ and since $d_i(y_i,z_i) \le r$. So $h_{q,i} := \psi_{q+1,i}^{-1}\phi_q : \Omega_q \to M$ is well defined for each q and all i large enough. From the C^∞ convergence $\phi_{q,i}^*g_i, \psi_{q,i}^*g_i \to g|_{\Omega_q}$, we also get the C^∞ convergence $h_{q,i}^*g \to g|_{\Omega_q}$, and moreover

$$\liminf_i d(x, h_{q,i}(x)) \ge \varepsilon\ , \qquad \limsup_i d(x, h_{q,i}(x)) \le r\ ,$$

because $\phi_{q,i}(x) = y_i$, $\psi_{q,i}(x) = z_i$ and $\varepsilon \le d_i(y_i,z_i) \le r$. Then, like in the proof of Lemma 5.6.1, an isometry $h : M \to M$ can be constructed so that $\varepsilon \le d(x,h(x)) \le r$, which contradicts (5.10). □

Lemma 5.6.3. *Let $n \in \mathbb{N}$ and $r > 0$. For any convergent sequence $[M_i,x_i] \to [M,x]$ in $\mathcal{M}^\infty_*(n)$ and each $y \in B(x,r)$, there are points $y_i \in B_i(x_i,r)$ such that $[M_i,y_i] \to [M,y]$ in $\mathcal{M}^\infty_*(n)$.*

Proof. Take a sequence of compact domains Ω_q of M such that $x,y \in \Omega_q$ and $d(x,\partial\Omega_q) \to \infty$ as $q \to \infty$. For each q, there is some index i_q such that, for each $i \ge i_q$ there is a C^∞ embedding $\phi_{q,i} : \Omega_q \to M_i$ satisfying $\phi_{q,i}(x) = x_i$ and $\phi_{q,i}^*g_i \to g|_{\Omega_q}$ as $i \to \infty$ with respect to the C^∞ topology. Let $y_{q,i} = \phi_{q,i}(y)$ for all $i \ge i_q$. Then, for each q and every $m \in \mathbb{Z}^+$, there is some index $i_{q,m} \ge i_q$ such that $d_i(x_i, y_{q,i}) < r$ and $\|\phi_{q,i}^*g_i - g\|_{C^m,\Omega_q,g} < 1/m$ for all $i \ge i_{q,m}$. Moreover we can assume that $i_{q,q} < i_{q+1,q+1}$ for all q. Now, let y_i be any point of $B_i(x_i,r)$ for $i < i_{0,0}$, and let $y_i = y_{q,i}$ for $i_{q,q} \le i < i_{q+1,q+1}$. Let us check that $[M_i,y_i] \to [M,y]$ in $\mathcal{M}^\infty_*(n)$. Fix any compact domain Ω of M containing y, and let $m \in \mathbb{N}$. We have $d(y,\partial\Omega_q) \to \infty$ as $q \to \infty$ because $d(x,\partial\Omega_q) \to \infty$ and $d(x,y) < r$. So there is some $q_0 \ge m$ such that $\Omega \subset \Omega_q$ for all $q \ge q_0$. For $i \ge i_{q_0,q_0}$, let $\phi_i = \phi_{q,i}|_\Omega$ if $i_{q,q} \le i < i_{q+1,q+1}$ with $q \ge q_0$. Then $\phi_i(y) = y_i$ and

$$\|\phi_i^*g_i - g\|_{C^m,\Omega_q,g} \le \|\phi_{q,i}^*g_i - g\|_{C^q,\Omega_q,g} < \frac{1}{q}$$

for $i_{q,q} \le i < i_{q+1,q+1}$, obtaining $\phi_i^*g_i \to g|_\Omega$ as $i \to \infty$. □

Lemma 5.6.4. *For $n \in \mathbb{N}$, let $[M,x] \in \mathcal{M}^\infty_*(n)$, and let $\mathcal{N}$ be a neighborhood of $[M,x]$ in $\mathcal{M}^\infty_*(n)$. Then there is some $\delta > 0$ and some neighborhood $\mathcal{L}$ of $[M,x]$ in $\mathcal{M}^\infty_*(n)$ such that $[L,z] \in \mathcal{N}$ for all $[L,y] \in \mathcal{L}$ and all $z \in B_L(y,\delta)$.*

Proof. There are some $m \in \mathbb{Z}^+$ and $\varepsilon > 0$, and a compact domain Ω of M containing x such that, for all $[L,z] \in \mathcal{M}^\infty_*(n)$, if there is some C^∞ embedding $\phi : \Omega \to L$ so that $\phi(x) = z$ and $\|\phi^*g_L - g_M\|_{C^m,\Omega,g_M} < \varepsilon$, then $[L,z] \in \mathcal{N}$. Take any compact domain Ω' of M whose interior contains Ω. There is some $\varepsilon_0 > 0$ and some neighborhood $\mathcal{H}$ of id_M in the group of diffeomorphisms of M with the weak C^m topology such that, for all $h \in \mathcal{H}$ and any metric tensor g' on Ω' satisfying $\|g' - g_M\|_{C^m,\Omega',g_M} < \varepsilon_0$, we have

$h(\Omega) \subset \Omega'$ and $\|h^*g' - g_M\|_{C^m,\Omega,g_M} < \varepsilon$. Moreover there is some $\delta' > 0$ such that, for each $z' \in B_M(x, \delta')$, there is some $h \in \mathcal{H}$ so that $h(x) = z'$. Let $\mathcal{L}$ be the neighborhood of $[M, x]$ in $\mathcal{M}_*^\infty(n)$ that consists of the points $[L, y] \in \mathcal{M}_*^\infty(n)$ such that there is some C^∞ embedding $\psi : \Omega' \to L$ so that $\psi(x) = y$ and $\|\psi^* g_L - g_M\|_{C^m,\Omega',g_M} < \varepsilon_0$. There is some $\delta > 0$ such that $B_L(y, \delta) \subset \psi(\Omega')$ and $\psi^{-1}(B_L(y, \delta)) \subset B_M(x, \delta')$ for all $[L, y] \in \mathcal{L}$ and $\psi : \Omega' \to L$ as above. Hence $z' = \psi^{-1}(z) \in B_M(x, \delta')$ for each $z \in B_L(y, \delta)$, and therefore there is some $h \in \mathcal{H}$ such that $h(x) = z'$. Then $\phi := \psi h$ is defined on Ω and satisfies $\phi(x) = \psi(z') = z$. Moreover

$$\|\phi^* g_L - g_M\|_{C^m,\Omega,g_M} = \|h^*\psi^* g_L - g_M\|_{C^m,\Omega,g_M} < \varepsilon$$

because $\|\psi^* g_L - g_M\|_{C^m,\Omega',g_M} < \varepsilon_0$ and $h \in \mathcal{H}$. □

5.7 Canonical bundles over $\mathcal{M}_{*,\mathrm{lnp}}^\infty(n)$

For each $n \in \mathbb{N}$, consider the set of pairs (M, ξ), where M is a complete connected Riemannian manifold without boundary of dimension n, and $\xi \in TM$. Like in the case of $\mathcal{M}_*(n)$, we can assume that the underlying set of each complete connected Riemannian n-manifold is contained in $\mathbb{R}$, obtaining that these pairs (M, ξ) form a well defined set. Define an equivalence relation on this set by declaring that (M, ξ) is equivalent to (N, ζ) if there is an isometric diffeomorphism $\phi : M \to N$ such that $\phi_*(\xi) = \zeta$. The class of a pair (M, ξ) will be denoted by $[M, \xi]$, and the corresponding set of equivalence classes will be denoted by $\mathcal{T}_*(n)$. If orthonormal tangent frames are used instead of tangent vectors in the above definition, we get a set denoted by $\mathcal{Q}_*(n)$. Let $\pi_{\mathcal{T}_*(n)} : \mathcal{T}_*(n) \to \mathcal{M}_*(n)$ and $\pi_{\mathcal{Q}_*(n)} : \mathcal{Q}_*(n) \to \mathcal{M}_*(n)$ be the maps defined by $\pi([M, \xi]) = [M, \pi_M(\xi)]$ and $\pi([M, f]) = [M, \pi_M(f)]$ for $[M, \xi] \in \mathcal{T}_*(n)$ and $[M, f] \in \mathcal{Q}_*(n)$; the simpler notation π will be used for $\pi_{\mathcal{T}_*(n)}$ and $\pi_{\mathcal{Q}_*(n)}$ if there is no danger of misunderstanding. For each $[M, x] \in \mathcal{M}_*(n)$, there are canonical surjections $T_xM \to \pi_{\mathcal{T}_*(n)}^{-1}([M, x])$, $\xi \mapsto [M, \xi]$, and $Q_xM \to \pi_{\mathcal{Q}_*(n)}^{-1}([M, x])$, $f \mapsto [M, f]$. Via the canonical surjection $Q_xM \to \pi_{\mathcal{Q}_*(n)}^{-1}([M, x])$, the canonical right action of $\mathrm{O}(n)$ on Q_xM induces a right action on $\pi_{\mathcal{Q}_*(n)}^{-1}([M, x])$; in this way, we get a canonical action of $\mathrm{O}(n)$ on $\mathcal{Q}_*(n)$ whose orbits are the fibers of $\pi_{\mathcal{Q}_*(n)}$. The operation of multiplication by scalars on T_xM also induces an action of $\mathbb{R}$ on $\pi_{\mathcal{T}_*(n)}^{-1}([M, x])$. However the sum operation of T_xM may not induce an operation on $\pi_{\mathcal{T}_*(n)}^{-1}([M, x])$. The following definition is analogous to Definition 1.3.1.

Definition 5.7.1. For each $m \in \mathbb{N}$, a sequence $[M_i, \xi_i] \in \mathcal{T}_*(n)$ (respectively, $[M_i, f_i] \in \mathcal{Q}_*(n)$) is said to be C^m *convergent* to $[M, \xi] \in \mathcal{T}_*(n)$ (respectively, $[M, f] \in \mathcal{Q}_*(n)$) if, with the notation $x = \pi(\xi)$ and $x_i = \pi_i(x_i)$ (respectively, $x = \pi(f)$ and $x_i = \pi_i(f_i)$), for each compact domain $\Omega \subset M$ containing x, there are pointed C^{m+1} embeddings

$\phi_i : (\Omega, x) \to (M_i, x_i)$ for large enough i such that $\phi_{i*}(\xi) = \xi_i$ (respectively, $\phi_{i*}(f) = f_i$), and $\phi_i^* g_i \to g|_\Omega$ as $i \to \infty$ with respect to the C^m topology. If $[M_i, \xi_i]$ (respectively, $[M_i, f_i]$) is C^m convergent to $[M, \xi]$ (respectively, $[M, f]$) for all m, then it is said that $[M_i, \xi_i]$ (respectively, $[M_i, f_i]$) is C^∞ *convergent* to $[M, \xi]$ (respectively, $[M, f]$).

Theorem 5.7.2. *The C^∞ convergence in $\mathcal{T}_*(n)$ and $\mathcal{Q}_*(n)$ describes a Polish topology.*

To prove Theorem 5.7.2, we follow the steps of Sections 5.3–5.5.

Definition 5.7.3. For $m \in \mathbb{N}$ and $R, r > 0$, let $V^m_{R,r}$ (respectively, $W^m_{R,r}$) be the set of pairs $([M, \xi], [N, \zeta]) \in \mathcal{T}_*(n) \times \mathcal{T}_*(n)$ (respectively, $([M, f], [N, h]) \in \mathcal{Q}_*(n) \times \mathcal{Q}_*(n)$) such that there is some (m, R, λ)-pointed local quasi-isometry $\phi : (M, x) \rightarrowtail (N, y)$ for some $\lambda \in [1, e^r)$ so that $\phi_*(\xi) = \zeta$ (respectively, $\phi_*(f) = h$).

The following proposition is proved like Proposition 5.3.2.

Proposition 5.7.4. *The following properties hold for all $m, m' \in \mathbb{N}$ and $R, S, r, s > 0$:*

(i) $(V^m_{e^r R,r})^{-1} \subset V^m_{R,r}$ *and* $(W^m_{e^r R,r})^{-1} \subset W^m_{R,r}$.

(ii) $V^{m_0}_{R_0,r_0} \subset V^m_{R,r} \cap V^{m'}_{S,s}$ *and* $W^{m_0}_{R_0,r_0} \subset W^m_{R,r} \cap W^{m'}_{S,s}$, *where* $m_0 = \max\{m, m'\}$, $R_0 = \max\{R, S\}$ *and* $r_0 = \min\{r, s\}$.

(iii) $\Delta \subset V^m_{R,r}$ *and* $\Delta \subset W^m_{R,r}$.

(iv) $V^m_{e^{r+s}R,r} \circ V^m_{e^{r+s}R,s} \subset V^m_{R,r+s}$ *and* $W^m_{e^{r+s}R,r} \circ W^m_{e^{r+s}R,s} \subset W^m_{R,r+s}$.

Proposition 5.7.5. $\bigcap_{R,r>0} V^m_{R,r} = \Delta$ *and* $\bigcap_{R,r>0} W^m_{R,r} = \Delta$ *for all* $m \in \mathbb{N}$.

Proof. We only prove the first equality because the proof of the second one is analogous. The inclusion "$\supset$" is obvious; thus let us prove "$\subset$". Let $([M, \xi], [N, \zeta]) \in \bigcap_{R,r>0} V^m_{R,r}$, and let $x = \pi_M(\xi)$ and $y = \pi_N(\zeta)$. Then there is a sequence of pointed local quasi-isometries $\phi_i : (M, x) \rightarrowtail (N, y)$, with corresponding types (m, R_i, λ_i), such that $\phi_{i*}(\xi) = \zeta$, and $R_i \uparrow \infty$ and $\lambda_i \downarrow 1$ as $i \to \infty$. According to the proof of Proposition 5.3.3, there is a pointed isometric immersion $\psi : (M, x) \to (N, y)$ so that, for any i, the restriction $\psi : B_M(x, R_i) \to N$ is the limit of the restrictions of a subsequence $\phi_{k(i,l)}$ in the weak C^m topology. Hence $\psi_*(\xi) = \lim_l \phi_{k(i,l)*}(\xi) = \zeta$, obtaining $[M, \xi] = [N, \zeta]$. □

By Propositions 5.7.4 and 5.7.5, the sets $V^m_{R,r}$ (respectively, $W^m_{R,r}$) form a base of entourages of a Hausdorff uniformity on $\mathcal{T}_*(n)$ (respectively, $\mathcal{Q}_*(n)$), which is also called the C^∞ *uniformity*. The corresponding topology is also called the C^∞ *topology*, and the corresponding space is denoted by $\mathcal{T}^\infty_*(n)$ (respectively, $\mathcal{Q}^\infty_*(n)$).

Remark 5.7.1. (i) The maps $\pi : \mathcal{T}_*^\infty(n) \to \mathcal{M}_*^\infty(n)$ and $\pi : \mathcal{Q}_*^\infty(n) \to \mathcal{M}_*^\infty(n)$ are uniformly continuous and open because $(\pi\times\pi)(V_{R,r}^m) = (\pi\times\pi)(W_{R,r}^m) = U_{R,r}^m$ for all $m \in \mathbb{N}$ and $R, r > 0$.

(ii) The canonical right $\mathrm{O}(n)$-action on $\mathcal{Q}_*^\infty(n)$ is continuous. This follows easily by using that the composite of maps is continuous in the weak C^∞ topology [44, p. 64, Exercise 10], and the following property that can be easily verified: for each $[M, f] \in \mathcal{Q}_*^\infty(n)$ and any neighborhood $\mathcal{N}$ of id_M in the space of C^∞ diffeomorphisms of M with the weak C^∞ topology, there is a neighborhood O of the identity element e in $\mathrm{O}(n)$ such that, for all $a \in O$, there is some $\phi \in \mathcal{N}$ so that $\phi(x) = x$ and $\phi_*(f) = h$.

Definition 5.7.6. For $R, r > 0$ and $m \in \mathbb{N}$, let $E_{R,r}^m$ (respectively, $F_{R,r}^m$) be the set of pairs $([M,\xi],[N,\zeta]) \in \mathcal{T}_*(n)\times\mathcal{T}_*(n)$ (respectively, $([M,f],[N,h]) \in \mathcal{Q}_*(n)\times\mathcal{Q}_*(n)$) such that, with the notation $x = \pi_M(\xi)$ and $y = \pi_N(\zeta)$, there is some C^{m+1} pointed local diffeomorphism $\phi\colon (M,x) \rightarrowtail (N,y)$ so that $\phi_*(\xi) = \zeta$ (respectively, $\phi_*(f) = h$), and $\|g_M - \phi^* g_N\|_{C^m,\Omega,g_M} < r$ for some compact domain $\Omega \subset \operatorname{dom}\phi$ with $B_M(x,R) \subset \Omega$.

Like in the case of relations on $\mathcal{M}_*(n)$, for $V \subset \mathcal{T}_*(n)\times\mathcal{T}_*(n)$, $W \subset \mathcal{Q}_*(n)\times\mathcal{Q}_*(n)$, $[M,\xi] \in \mathcal{T}_*(n)$ and $[M,f] \in \mathcal{Q}_*(n)$, the simpler notation $V(M,\xi)$ and $W(M,f)$ is used instead of $V([M,\xi])$ and $W([M,f])$.

Remark 5.7.2. By (4.3), a sequence $[M_i,\xi_i] \in \mathcal{T}_*(n)$ (respectively, $[M_i,f_i] \in \mathcal{Q}_*(n)$) is C^∞ convergent to $[M,\xi] \in \mathcal{T}_*(n)$ (respectively, $[M,f] \in \mathcal{Q}_*(n)$) if and only if it is eventually in $E_{R,r}^m(M,\xi)$ (respectively, $F_{R,r}^m(M,f)$) for arbitrary $m \in \mathbb{N}$ and $R, r > 0$.

Proposition 5.7.7. (i) *For $R, r > 0$, if $0 < \varepsilon \le \min\{1-e^{-2r}, e^{2r}-1\}$, then $E_{R,\varepsilon}^0 \subset V_{R,r}^0$ and $F_{R,\varepsilon}^0 \subset W_{R,r}^0$.*

(ii) *For all $m \in \mathbb{Z}^+$, $R, r > 0$ and $[M,\xi] \in \mathcal{T}_*(n)$ (respectively, $[M,f] \in \mathcal{P}_*(n)$), there is some $\varepsilon > 0$ such that $E_{R,\varepsilon}^m(M,\xi) \subset V_{R,r}^m(M,\xi)$ (respectively, $F_{R,\varepsilon}^m(M,\xi) \subset W_{R,r}^m(M,\xi)$).*

Proof. Let us show (i) for the case of $V_{R,r}^0$, since the case of $W_{R,r}^0$ is analogous. Let $([M,\xi],[N,\zeta]) \in E_{R,\varepsilon}^0$, and let $x = \pi_M(\xi)$ and $y = \pi_N(\zeta)$. Then there is a C^1 pointed local diffeomorphism $\phi : (M,x) \rightarrowtail (N,y)$ such that $\phi_*(\xi) = \zeta$, and $\varepsilon_0 := \|g_M - \phi^* g_N\|_{C^0,\Omega,g_M} < \varepsilon$ for some compact domain $\Omega \subset \operatorname{dom}\phi$ with $B_M(x,R) \subset \Omega$. According to the proof of Proposition 5.4.4-(i), ϕ is a $(0,R,\lambda)$-pointed local quasi-isometry if $1 \le \lambda < e^r$ and $\varepsilon_0 \le \min\{1-\lambda^{-2}, \lambda^2-1\}$, obtaining that $([M,\xi],[N,\zeta]) \in V_{R,r}^0$.

As above, let us prove (ii) only for the case of $V_{R,r}^m(M,\xi)$. Take $m \in \mathbb{Z}^+$, $R, r > 0$ and $[M,\xi],[N,\zeta] \in \mathcal{T}_*(n)$, and let $x = \pi_M(\xi)$ and $y = \pi_N(\zeta)$. According to the proof of Proposition 5.4.4-(ii), there is some $\varepsilon > 0$ such that, for every C^{m+1} pointed local diffeomorphism $\phi\colon (M,x) \rightarrowtail (N,y)$, if $\|g_M - \phi^* g_N\|_{C^m,\Omega,g_M} < \varepsilon$ for some compact domain $\Omega \subset \operatorname{dom}\phi \cap \operatorname{Int}(K)$ with $B_M(x,R) \subset \Omega$, then ϕ is an (m,R,λ)-pointed local quasi-isometry $(M,x) \rightarrowtail (N,y)$ for some $\lambda \in [1,e^r)$. Therefore $[N,\zeta] \in V^m_{R,r}(M,\xi)$ if $[N,\zeta] \in E^m_{R,\varepsilon}(M,\xi)$. □

Proposition 5.7.8. (i) *For all $R,r > 0$, if $e^{2\varepsilon} - e^{-2\varepsilon} \le r$, then $V^0_{R,\varepsilon} \subset E^0_{R,r}$ and $W^0_{R,\varepsilon} \subset F^0_{R,r}$.*

(ii) *For all $m \in \mathbb{Z}^+$, $R,r > 0$ and $[M,\xi] \in \mathcal{T}_*(n)$ (respectively, $[M,f] \in \mathcal{Q}_*(n)$), there is some $\varepsilon > 0$ such that $V^m_{R,\varepsilon}(M,\xi) \subset E^m_{R,r}(M,\xi)$ (respectively, $W^m_{R,\varepsilon}(M,f) \subset F^m_{R,r}(Mf)$).*

Proof. This result follows from the proof of Proposition 5.4.5 in the same way as Proposition 5.7.7 follows from Proposition 5.4.4. □

As a direct consequence of Remark 5.7.2, and Propositions 5.7.7 and 5.7.8, we get that the C^∞ convergence in $\mathcal{T}_*(n)$ and $\mathcal{Q}_*(n)$ describes the C^∞ topology.

Proposition 5.7.9. *$\mathcal{T}^\infty_*(n)$ and $\mathcal{Q}^\infty_*(n)$ are separable*

Proof. With the notation of Proposition 5.5.1, for every $M \in \mathcal{C}$, let $\mathcal{D}'_M$ and $\mathcal{D}''_M$ be countable dense subsets of TM and QM, respectively. Then the countable sets

$$\{\, [(M,g),\xi] \mid M \in \mathcal{C},\ g \in \mathcal{G}_M,\ \xi \in \mathcal{D}'_M \,\} \quad \text{and} \quad \{\, [(M,g),f] \mid M \in \mathcal{C},\ g \in \mathcal{G}_M,\ f \in \mathcal{D}''_M \,\}$$

are dense in $\mathcal{T}^\infty_*(n)$ and $\mathcal{Q}^\infty_*(n)$, respectively. □

Proposition 5.7.10. *$\mathcal{T}^\infty_*(n)$ and $\mathcal{Q}^\infty_*(n)$ are completely metrizable*

Proof. Only the case of $\mathcal{T}^\infty_*(n)$ is proved, the other case being similar. The C^∞ uniformity on $\mathcal{T}^\infty_*(n)$ is metrizable because it has a countable base of entourages. Thus it is enough to check that this uniformity is complete.

Consider an arbitrary Cauchy sequence $[M_i,\xi_i]$ in $\mathcal{T}_*(n)$ with respect to the C^∞ uniformity, and let $x_i = \pi_i(\xi_i) \in M_i$. We have to prove that $[M_i,\xi_i]$ is convergent in $\mathcal{T}^\infty_*(n)$. By taking a subsequence if necessary, we can suppose that $([M_i,\xi_i],[M_{i+1},\xi_{i+1}]) \in V^{m_i}_{R_i,r_i}$ for sequences m_i, and R_i and r_i satisfying the conditions of the proof of Proposition 5.5.2. Thus, for each i, there is some $\lambda_i \in (1,e^{r_i})$ and some (m_i,R_i,λ_i)-pointed local quasi-isometry $\phi_i\colon (M_i,x_i) \rightarrowtail (M_{i+1},x_{i+1})$, which can be assumed to be C^∞ (Remark 5.2.1-(iii)), such that $\phi_{i*}(\xi_i) = \xi_{i+1}$. Then, with the notation of the proof of Proposition 5.5.2, we have $\psi_{ij*}(\xi_i) = \xi_j$ for $i < j$. Therefore there is some $\hat\xi \in T_{\hat x}\widehat{M}$ so that $\psi_{i*}(\xi_i) = \hat\xi$ for all i, obtaining that $([M_i,\xi_i],[\widehat{M},\hat\xi]) \in U^{m_i}_{R'_i/\bar\lambda_i,s_i}$ for all i according to the proof of Proposition 5.5.2. Hence $[M_i,\xi_i] \to [\widehat{M},\hat\xi]$ as $i \to \infty$ in $\mathcal{T}^\infty_*(n)$. □

Propositions 5.7.9 and 5.7.10 together mean that $\mathcal{T}_*^\infty(n)$ and $\mathcal{Q}_*^\infty(n)$ are Polish, completing the proof of Theorem 5.7.2.

Let $\mathcal{T}_{*,\mathrm{lnp}}^\infty(n) \subset \mathcal{T}_*^\infty(n)$ and $\mathcal{Q}_{*,\mathrm{lnp}}^\infty(n) \subset \mathcal{Q}_*^\infty(n)$ be the subspaces defined by locally non-periodic manifolds.

Proposition 5.7.11. (i) *The projection* $\pi : \mathcal{T}_{*,\mathrm{lnp}}^\infty(n) \to \mathcal{M}_{*,\mathrm{lnp}}^\infty(n)$ *admits the structure of a Riemannian vector bundle of rank* n *so that the canonical map* $T_xM \to \pi^{-1}([M,x])$ *is a orthogonal isomorphism for each* $[M,x] \in \mathcal{M}_{*,\mathrm{lnp}}^\infty(n)$.

(ii) *The projection* $\pi : \mathcal{Q}_{*,\mathrm{lnp}}^\infty(n) \to \mathcal{M}_{*,\mathrm{lnp}}^\infty(n)$ *admits the structure of a* $\mathrm{O}(n)$*-principal bundle canonically isomorphic to the* $\mathrm{O}(n)$*-principal bundle of orthonormal references of* $\mathcal{T}_{*,\mathrm{lnp}}^\infty(n)$.

Proof. Obviously, the canonical $\mathrm{O}(n)$-action on $\mathcal{Q}_*^\infty(n)$ preserves $\mathcal{Q}_{*,\mathrm{lnp}}^\infty(n)$, and the $\mathrm{O}(n)$-orbits in $\mathcal{Q}_{*,\mathrm{lnp}}^\infty(n)$ are the fibers of $\pi : \mathcal{Q}_{*,\mathrm{lnp}}^\infty(n) \to \mathcal{M}_{*,\mathrm{lnp}}^\infty(n)$.

Claim 5.7.1. For all $[M,x] \in \mathcal{M}_{*,\mathrm{lnp}}^\infty(n)$, the canonical maps $T_xM \to \pi_{\mathcal{T}_*(n)}^{-1}([M,x])$ and $Q_xM \to \pi_{\mathcal{Q}_*(n)}^{-1}([M,x])$ are bijections.

Let us show the case of the first map in Claim 5.7.1, the case of the second one being similar. It was already pointed out that the canonical map $T_xM \to \pi_{\mathcal{T}_*(n)}^{-1}([M,x])$ is surjective, and let us to prove that it is also injective. If $[M,\xi] = [M,\zeta]$ for some $\xi,\zeta \in T_xM$, then $\phi_*(\xi) = \zeta$ for some $\phi \in \mathrm{Iso}(M)$ with $\phi(x) = x$. But $\phi = \mathrm{id}_M$ because M is locally non-periodic, obtaining $\xi = \zeta$.

Let X be a completely regular space with a right action of a Lie group G, and let $G_x \subset G$ denote the isotropy subgroup at some point $x \in X$. Recall that a *slice* at x is a subspace $S \subset X$ containing x such that $S \cdot G$ is open in X, and there is a G-equivariant continuous map $\kappa : S\cdot G \to G_x\backslash G$ with $\kappa^{-1}(G_x) = S$ [58, Definition 2.1.1]. Since $\mathcal{Q}_{*,\mathrm{lnp}}^\infty(n)$ is completely regular and $\mathrm{O}(n)$ is compact, the $\mathrm{O}(n)$-action on $\mathcal{Q}_{*,\mathrm{lnp}}^\infty(n)$ has a slice $\mathcal{S}$ at each point $[M,f] \in \mathcal{Q}_{*,\mathrm{lnp}}^\infty(n)$ [58, Theorem 2.3.3] (see also [45], [61, Theorems 5.1 and 5.2] and [18, Theorems 11.3.9 and 11.3.14]). Then $\Theta := \pi(\mathcal{S}) = \pi(\mathcal{S}\cdot\mathrm{O}(n))$ is open in $\mathcal{M}_{*,\mathrm{lnp}}^\infty(n)$ by Remark 5.7.1-(i).

Claim 5.7.2. $\pi : \mathcal{S} \to \Theta$ is a homeomorphism.

This is the restriction of a continuous map (Remark 5.7.1-(i)), and therefore it is continuous. This map is also open because, for every open $W \subset \mathcal{S}$, the set $W \cdot \mathrm{O}(n)$ is open in $\mathcal{Q}_{*,\mathrm{lnp}}^\infty(n)$ [58, Corollary of Proposition 2.1.2], and thus $\pi(W) = \pi(W \cdot \mathrm{O}(n))$ is open in $\mathcal{M}_*^\infty(n)$ (Remark 5.7.1-(i)). Obviously, $\pi : \mathcal{S} \to \Theta$ is surjective, and let us show that it is also injective. Take $[N,p],[L,q] \in \mathcal{S}$ such that $\pi([N,p]) = \pi([N,q]) =: x$. Thus there is some $a \in \mathrm{O}(n)$ so that $[L,q] = [N,p] \cdot a$. Since the isotropy group at $[M,f]$ is trivial by Claim 5.7.1, there is an $\mathrm{O}(n)$-equivariant continuous map $\kappa : \mathcal{S}\cdot\mathrm{O}(n) \to \mathrm{O}(n)$ so that $\kappa^{-1}(e) = \mathcal{S}$. It follows that $e = \kappa([L,q]) = \kappa([N,p]\cdot a) = \kappa([N,p])\,a = a$, obtaining $[L,q] = [N,p]$, which completes the proof of Claim 5.7.2.

According to Claim 5.7.2, the inverse of $\pi : \mathcal{S} \to \Theta$ defines a continuous local section $\sigma : \Theta \to \mathcal{Q}^\infty_{*,\mathrm{lnp}}(n)$ of $\pi : \mathcal{Q}^\infty_{*,\mathrm{lnp}}(n) \to \mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$. By the existence of continuous local sections, and since the $\mathrm{O}(n)$-action on $\mathcal{Q}^\infty_{*,\mathrm{lnp}}(n)$ is continuous and free (Remark 5.7.1-(ii) and Claim 5.7.1), it easily follows that $\pi : \mathcal{Q}^\infty_{*,\mathrm{lnp}}(n) \to \mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$ admits the structure of an $\mathrm{O}(n)$-principal bundle.

By Claim 5.7.1, $\pi^{-1}_{\mathcal{T}_*(n)}([M,x])$ canonically becomes an orthogonal vector space for each $[M,x] \in \mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$, and we can canonically identify $\pi_{\mathcal{Q}_*^{-1}(n)}([M,x])$ to the set of linear isometries $\pi^{-1}_{\mathcal{T}_*(n)}([M,x]) \to \mathbb{R}^n$. The continuity of the mapping $([M,f],[M,\xi]) \mapsto [M,f]([M,\xi])$ is easy to check. By using this identity, we get a homeomorphism $\theta : \pi^{-1}_{\mathcal{T}_*(n)}(\Theta) \to \mathbb{R}^n \times \Theta$ defined by $\theta([M,\xi]) = (\sigma([M,x])([M,\xi]),[M,x])$, where $\pi([M,\xi]) = [M,x]$, whose inverse map is given by $\theta^{-1}(v,[M,x]) = [M,\sigma([M,x])^{-1}(v)]$. If $\sigma' : \Theta' \to \mathcal{Q}^\infty_{*,\mathrm{lnp}}(n)$ is another local section of $\pi : \mathcal{Q}^\infty_{*,\mathrm{lnp}}(n) \to \mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$ defining a map $\theta' : \pi^{-1}(\Theta') \to \mathbb{R}^n \times \Theta'$ as above, and $[M,x] \in \Theta \cap \Theta'$, then the composite

$$\mathbb{R}^n \equiv \mathbb{R}^n \times \{[M,x]\} \xrightarrow{\theta^{-1}} \pi^{-1}_{\mathcal{T}_*(n)}([M,x]) \xrightarrow{\theta'} \mathbb{R}^n \times \{[M,x]\} \equiv \mathbb{R}^n$$

is the orthogonal isomorphism $\sigma'([M,x]) \circ \sigma([M,x])^{-1}$. It follows that $\pi : \mathcal{T}^\infty_{*,\mathrm{lnp}}(n) \to \mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$, with these local trivializations, becomes an orthogonal vector bundle of rank n so that the canonical map $T_xM \to \pi^{-1}([M,x])$ is a orthogonal isomorphism for all $[M,x] \in \mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$. Moreover, by Claim 5.7.1, there is a canonical isomorphism between $\mathcal{Q}^\infty_{*,\mathrm{lnp}}(n)$ and the $\mathrm{O}(n)$-principal bundle of orthonormal frames of $\mathcal{T}^\infty_{*,\mathrm{lnp}}(n)$. □

By the compatibility of exponential maps and isometries, a map $\exp : \mathcal{T}^\infty_*(n) \to \mathcal{M}^\infty_*(n)$ is well defined by setting $\exp([M,\xi]) = [M,\exp_M(\xi)]$. For each $[M,x] \in \mathcal{M}^\infty_*(n)$, the restriction $\exp : \pi^{-1}([M,x]) \to \mathcal{M}^\infty_*(n)$ may be denoted by $\exp_{[M,x]}$.

Lemma 5.7.12. *Consider convergent sequences $[M_i,f_i] \to [M,f]$ and $[M_i,f'_i] \to [M,f']$ in $\mathcal{Q}^\infty_*(n)$ for some $n \in \mathbb{Z}^+$. Let $x = \pi(f)$, $x' = \pi(f')$, $x_i = \pi_i(f_i)$ and $x'_i = \pi_i(f'_i)$. Suppose that there is some $r > 0$ such that*

$$\left\{h \in \mathrm{Iso}(M) \mid h(x) \in \overline{B}(x,2r)\right\} = \{\mathrm{id}_M\}\ , \tag{5.11}$$

and $d(x,x'), d_i(x_i,x'_i) \le r$ for all i. Then there is some compact domain Ω in M whose interior contains x and x', and there are C^∞ embeddings $\phi_i : \Omega \to M_i$ for i large enough so that $\phi_{i*}(f) = f_i$ and $\lim_i \phi^{-1}_{i*}(f'_i) = f'$ in PM, and $\lim_i \phi_i^* g_i = g|_\Omega$ with respect to the C^∞ topology.

Proof. Let Ω_q be a sequence of compact domains in M such that

$$\overline{B}(x,r) \subset \operatorname{Int}(\Omega_q) , \quad \operatorname{Pen}(\Omega_q, \operatorname{diam}(\Omega_q)) \subset \operatorname{Int}(\Omega_{q+1}) ;$$

in particular, $x' \in \operatorname{Int}(\Omega_q)$. By the convergence $[M_i, f_i] \to [M, f]$ and $[M_i, f_i'] \to [M, f']$ in $\mathcal{Q}_*^\infty(n)$, for each q, there are C^∞ embeddings $\phi_{q,i}, \psi_{q,i} : \Omega_q \to M_i$ for i large enough so that $\phi_{q,i*}(f) = f_i$, $\psi_{q,i*}(f') = f_i'$, and $\lim_i \phi_{q,i}^* g_i = g|_{\Omega_q}$ and $\lim_i \psi_{q,i}^* g_i = g|_{\Omega_q}$ with respect to the C^∞ topology; in particular, $\phi_{q,i}(x) = x_i$ and $\psi_{q,i}(x') = x_i'$. We have $x_i' \in \overline{B}_i(x_i, r) \subset \operatorname{Int}(\phi_{q,i}(\Omega_q))$ for i large enough, depending on q, and therefore $\phi_{q,i}(\Omega_q) \cap \psi_{q,i}(\Omega_q) \neq \emptyset$. Hence

$$\psi_{q,i}(\Omega_q) \subset \operatorname{Pen}_i(\phi_{q,i}(\Omega_q), \operatorname{diam}(\phi_{q,i}(\Omega_q))) \subset \operatorname{Int}(\phi_{q,i}(\Omega_{q+1}))$$

for i large enough, depending on q. It follows that $h_{q,i} := \phi_{q+1,i}^{-1} \psi_{q,i}$ is a well defined C^∞ embedding $\Omega_q \to M$. Observe that $\lim_i h_{q,i}^* g = g|_{\Omega_q}$ with respect to the C^∞ topology. Moreover

$$\begin{aligned}\limsup_i d(x, h_{q,i}(x)) &= \limsup_i d(x, \phi_{q+1,i}^{-1} \psi_{q,i}(x)) = \limsup_i d_i(x_i, \psi_{q,i}(x)) \\ &\leq \limsup_i d_i(x_i, x_i') + \limsup_i d_i(x_i', \psi_{q,i}(x)) \leq r + d(x', x) \leq 2r .\end{aligned}$$

If the statement is not true, then some neighborhood U of f' in PM contains no accumulation point of the sequence $\phi_{q+1,i*}^{-1}(f_i') = \phi_{q+1,i*}^{-1} \psi_{q,i*}(f') = h_{q,i*}(f')$ for each q. With the arguments of the proof of Lemma 5.6.1, it follows that there is some $h \in \operatorname{Iso}(M)$ such that $d(x, h(x)) \leq 2r$ and $h_*(f') \notin U$, which contradicts (5.11). □

5.8 Center of mass

The main tool used to prove Theorem 1.3.3-(ii)–(v) is the Riemannian center of mass of a mass distribution on a Riemannian manifold M [50], [24, Section IX.7]; especially, we will use the continuous dependence of the center of mass on the mass distribution and the metric tensor.

Recall that a domain $\Omega \subset M$ is said to be *convex* when, for all $x, y \in \Omega$, there is a unique minimizing geodesic segment from x to y in M that lies in Ω (see e.g. [24, Section IX.6]). For example, sufficiently small balls are convex. For a fixed convex compact domain Ω in M, let $\mathcal{C}(\Omega)$ be the set of functions $f \in C^2(\Omega)$ such that the gradient $\operatorname{grad} f$ is an outward pointing vector field on $\partial\Omega$ and $\operatorname{Hess} f$ is positive definite on the interior $\operatorname{Int}(\Omega)$ of Ω. Notice that $\mathcal{C}(\Omega)$ is open in the Banach space $C^2(\Omega)$ with the norm $\|\ \|_{C^2,\Omega,g}$, and thus it is a C^∞ Banach manifold. Moreover $\mathcal{C}(\Omega)$ is preserved by the operations of sum and product by positive numbers. Any $f \in \mathcal{C}(\Omega)$ attains its minimum value at a unique point $(f) \in \operatorname{Int}(\Omega)$, defining a function $: \mathcal{C}(\Omega) \to \operatorname{Int}(\Omega)$.

Lemma 5.8.1. *is continuous.*

Proof. Consider the map $\mathbf{v} : \mathcal{C}(\Omega) \times \mathrm{Int}(\Omega) \to T\Omega$ defined by $\mathbf{v}(f, x) = \operatorname{grad} f(x)$, and let $Z \subset T\Omega$ denote the image of the zero section. Since the graph of is equal to $\mathbf{v}^{-1}(Z)$, it is enough to prove the following.

Claim 5.8.1. $\mathbf{v}$ is C^1 and transverse to Z.

Here, smoothness and transversality refer to $\mathbf{v}$ considered as a map between C^∞ Banach manifolds [1, p. 45].

Let $\pi_{\mathcal{H}}$ and $\pi_{\mathcal{V}}$ denote the orthogonal projections of $T^{(2)}\Omega$ onto $\mathcal{H}$ and $\mathcal{V}$, respectively. Let $\mathfrak{X}^1(\Omega)$ denote the Banach space of C^1 vector fields over Ω with the norm $\| \ \|_{C^1,\Omega,g}$, which is equivalent to the norm $\| \ \|_1$ defined by

$$\|X\|_1 = \sup\{\, |X(x)| + |\nabla X(x)| \mid x \in \Omega \,\} \ .$$

The gradient map, $\operatorname{grad} : C^2(\Omega) \to \mathfrak{X}^1(\Omega)$, is a continuous linear map between Banach spaces, and therefore it is C^∞. The evaluation map, $\mathrm{ev} : \mathfrak{X}^1(\Omega) \times \Omega \to T\Omega$, is C^1 because, if $X \in \mathfrak{X}^1(\Omega)$, $Y \in T_X\mathfrak{X}^1(\Omega) \equiv \mathfrak{X}^1(\Omega)$, $x \in \Omega$ and $\xi \in T_x\Omega$, then $\mathrm{ev}_*(Y, \xi) \in T_\xi T\Omega$ is easily seen to be determined by the conditions $\pi_{\mathcal{H}}(\mathrm{ev}_*(Y, \xi)) \equiv \xi$ in $\mathcal{H}_\xi \equiv T_x\Omega$ and $\pi_{\mathcal{V}}(\mathrm{ev}_*(Y, \xi)) \equiv Y(x) + \nabla_\xi X$ in $\mathcal{V}_\xi \equiv T_x\Omega$. Therefore $\mathbf{v}$ is C^1 because it is the restriction to $\mathcal{C}(\Omega) \times \mathrm{Int}(\Omega)$ of the composition

$$C^2(\Omega) \times \Omega \xrightarrow{\operatorname{grad} \times \mathrm{id}_\Omega} \mathfrak{X}^1(\Omega) \times \Omega \xrightarrow{\mathrm{ev}} T\Omega \ .$$

Fix any $f \in \mathcal{C}(\Omega)$ and $x \in \mathrm{Int}(\Omega)$ with $\mathbf{v}(f, x) \in Z$; thus $\operatorname{grad} f(x) = 0_x$.

Claim 5.8.2. $\pi_{\mathcal{V}} : \mathbf{v}_*(\{0_f\} \times T_x\Omega) \to \mathcal{V}_{0_x}$ is an isomorphism.

For any $\xi \in T_x\Omega$,

$$\pi_{\mathcal{V}}\, \mathbf{v}_*(0_f, \xi) = \pi_{\mathcal{V}}\, (\operatorname{grad} f)_*(\xi) \equiv \nabla_\xi \operatorname{grad} f$$

in $\mathcal{V}_{0_x} \equiv T_x\Omega$. Then Claim 5.8.2 follows because the mapping $\xi \mapsto \nabla_\xi \operatorname{grad} f$ is an automorphism of $T_x\Omega$ since $\operatorname{Hess} f$ is positive definite at x and $\operatorname{Hess} f(\xi, \cdot) = g(\nabla_\xi \operatorname{grad} f, \cdot)$ on T_xM.

From Claim 5.8.2, it follows that $\mathbf{v}_*(\{0_f\} \times T_x\Omega)$ is a linear complement to $\mathcal{H}_{0_x} = T_{0_x}Z$ in $T_{0_x}T\Omega$; in particular, it is closed in $T_{0_x}T\Omega$ because $T_{0_x}T\Omega$ is Hausdorff of finite dimension.

Since $\mathbf{v}_* : T_f\mathcal{C}(\Omega) \times T_x\Omega \to T_{0_x}T\Omega$ is linear and continuous, and $T_{0_x}T\Omega$ is Hausdorff of finite dimension, we get that the space $\left(\mathbf{v}_{*(f,x)}\right)^{-1}(T_{0_x}Z)$ is closed and of finite codimension in the Banach space $T_f\mathcal{C}(\Omega) \times T_x\Omega$, and therefore it has a closed linear complement in $T_f\mathcal{C}(\Omega) \times T_x\Omega$ (see e.g. [65, p. 22]), which completes the proof of Claim 5.8.1. $\square$

Remark 5.8.1. (i) In the last part of the above proof, the space $\left(\mathbf{v}_{*(f,x)}\right)^{-1}(T_{0_x}Z)$ can be described as follows. Since $h \mapsto \operatorname{grad} h(x)$ defines a continuous linear map $C^2(\Omega) \to T_x\Omega$, we have $\mathbf{v}_*(T_f\mathcal{C}(\Omega) \times \{0_x\}) \subset \mathcal{V}_{0_x}$ and $\mathbf{v}_*(h, 0_x) \equiv \operatorname{grad} h(x)$ in $\mathcal{V}_{0_x} \equiv T_x\Omega$ for any $h \in C^2(\Omega) \equiv T_f\mathcal{C}(\Omega)$, giving

$$\left(\mathbf{v}_{*(f,x)}\right)^{-1}(T_{0_x}Z) \equiv \{\,(h,\xi) \in C^2(\Omega) \times T_x\Omega \mid \operatorname{grad} h(x) + \nabla_\xi \operatorname{grad} f = 0\,\}\ ,$$

which is obviously closed and of finite codimension in $C^2(\Omega) \times T_x\Omega$.

(ii) In Lemma 5.8.1, the map is C^m if the Banach space $C^{m+2}(\Omega)$ is used instead of $C^2(\Omega)$.

Suppose that the Riemannian manifold M is connected and complete. Let (A,μ) be a probability space, B a convex open ball of radius $r > 0$ in M, and $f : A \to B$ a measurable map, which is called a *mass distribution* on B. Consider the C^∞ function $P_f : B \to \mathbb{R}$ defined by

$$P_f(x) = \frac{1}{2}\int_A d(x, f(a))^2\,\mu(a)\ .$$

Proposition 5.8.2 (H. Karcher [50, Theorem 1.2]). *With the above notation and conditions, the following properties hold:*

(i) $\operatorname{grad} P_f$ *is an outward pointing vector field on the boundary* $\partial\overline{B}$.

(ii) *If* $\delta > 0$ *is an upper bound for the sectional curvatures of* M *in* B, *and* $2r < \pi/2\sqrt{\delta}$, *then* $\operatorname{Hess} P_f$ *is positive definite on* B.

If the hypotheses of Proposition 5.8.2 are satisfied, then $P_f \in \mathcal{C}(\overline{B})$, and therefore P_f reaches its minimum on $\overline{B}$ at a unique point $\mathcal{C}_f \in B$, which is called the *center of mass* of f. It is known that $\mathcal{C}_f$ depends continuously on f with respect to the supremum distance when (A,μ) is fixed [50, Corollary 1.6]; indeed, the following result follows directly from Lemma 5.8.1.

Corollary 5.8.3. (i) $\mathcal{C}_f$ *depends continuously on* f *and the metric tensor of* M.

(ii) *If* A *is the Borel* σ*-algebra of a metric space, then* $\mathcal{C}_f$ *depends continuously on* μ *in the weak-$*$ topology.*

5.9 Foliated structure of $\mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$

The goal of this section is to prove Theorem 1.3.3-(ii)–(v).

For any point $[M,x] \in \mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$, choose some $r, \varepsilon > 0$ and some neighborhood $\mathcal{N}_0$ of $[M,x]$ in $\mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$ satisfying the statement of Lemma 5.6.2 with $\varepsilon \le r/5$. Using [60, Chapter 6, Theorem 3.6], we can assume that ε and $\mathcal{N}_0$ are so small that $B_L(y,\varepsilon)$ satisfies the conditions of Proposition 5.8.2 in L for all $[L,y]\in\mathcal{N}_0$. Take any continuous function $\lambda:\mathcal{M}^{\infty}_{*}(n)\to[0,1]$ supported in $\mathcal{N}_0$ and with $\lambda([M,x])=1$, whose existence is a simple consequence of the metrizability of $\mathcal{M}^{\infty}_{*}(n)$ (Theorem 1.3.2). For $[L,y]\in\mathcal{N}_0$, let ω_L denote the Riemannian density of L, and let $\lambda_{L,y}:L\to[0,1]$ be the function defined by

$$\lambda_{L,y}(z)=\begin{cases}\lambda([L,z]) & \text{if } d_L(y,z)\le\varepsilon\\ 0 & \text{if } d_L(y,z)\ge\varepsilon\ ,\end{cases}$$

which is well defined and continuous by Lemma 5.6.2. Take another neighborhood $\mathcal{N}\subset\mathcal{N}_0$ of $[M,x]$ where $\lambda>0$. For $[L,y]\in\mathcal{N}$, we have $\int_L\lambda_{L,y}\,\omega_L>0$, and set

$$\bar\lambda_{L,y}=\frac{\lambda_{L,y}}{\int_L\lambda_{L,y}\,\omega_L}\ .$$

Then $\mu_{L,y}=\bar\lambda_{L,y}\,\omega_L$ is a continuous density defining a probability measure on L, and the identity map $(L,\mu_{L,y})\to L$ is a distribution of mass on L satisfying the conditions of Proposition 5.8.2 with $B_L(y,\varepsilon)$. Thus its center of mass, $\mathcal{C}_{L,y}$, is defined in $B_L(y,\varepsilon)$. Let $\mathbf{c}:\mathcal{N}\to\mathcal{M}^{\infty}_{*}(n)$ be the map given by $\mathbf{c}([L,y])=[L,\mathcal{C}_{L,y}]$.

Lemma 5.9.1. *If $[L,y],[L,y']\in\mathcal{N}$ and $d_L(y,y')\le\varepsilon$, then $\mathbf{c}([L,y])=\mathbf{c}([L,y'])$.*

Proof. Take any point $z\in L$. If $[L,z]\notin\mathcal{N}_0$ or $d_L(y,z),d_L(y',z)>\varepsilon$, then $\lambda_{L,y}(z)=\lambda_{L,y'}(z)=0$. If $[L,z]\in\mathcal{N}_0$ and $d_L(y,z)\le\varepsilon$, then $d_L(y',z)\le2\varepsilon$, obtaining $d_L(y',z)\le\varepsilon$ by Lemma 5.6.2 since $5\varepsilon\le r$, and therefore $\lambda_{L,y}(z)=\lambda_{L,y'}(z)=\lambda([L,z])$. If $[L,z]\in\mathcal{N}_0$ and $d_L(y',z)\le\varepsilon$, we similarly get $\lambda_{L,y}(z)=\lambda_{L,y'}(z)$. Thus $\lambda_{L,y}=\lambda_{L,y'}$, obtaining $\mathcal{C}_{L,y}=\mathcal{C}_{L,y'}$, and therefore $\mathbf{c}([L,y])=\mathbf{c}([L,y'])$. □

Lemma 5.9.2. $\mathbf{c}$ *is continuous.*

Proof. Take any convergent sequence $[L_i,y_i]\to[L,y]$ in $\mathcal{N}$. Let Ω be a compact domain in L whose interior contains $\overline{B}_L(y,\varepsilon)$. Then there is a C^∞ embedding $\phi_i:\Omega\to L_i$ for each i large enough so that $\lim_i\phi_i^*g_i=g|_\Omega$ with respect to the C^∞ topology. It follows that $\lim_i\phi_i^*\mu_{L_i,y_i}=\mu_{L,y}|_\Omega$ with respect to the C^0 topology by the continuity of λ, and thus this convergence also holds in the space of probability measures on Ω with the weak-$*$ topology. Since $\phi_i^{-1}(\mathcal{C}_{L_i,y_i})$ is the center of mass of the mass distribution on Ω defined by the probability measure $\phi_i^*\mu_{L_i,y_i}$, it follows from Corollary 5.8.3 that $\lim_i\phi_i^{-1}(\mathcal{C}_{L_i,y_i})=\mathcal{C}_{L,y}$ in L. Therefore $\lim_i\mathbf{c}([L_i,y_i])=\mathbf{c}([L,y])$ in $\mathcal{M}^{\infty}_{*}(n)$ because Ω is arbitrary. □

Let $\mathcal{Z} = \mathbf{c}(\mathcal{N})$, and let $\mathcal{N}' = \bigcup_{[L,c]\in\mathcal{Z}} \iota_L(B_L(c,\varepsilon))$, which contains $\mathcal{N}$ since $d_M(y,\mathcal{C}_{L,y}) < \varepsilon$ for all $[L,y] \in \mathcal{N}$. Also, let $\mathbf{c}' : \mathcal{N}' \to \mathcal{Z}$ be defined by the condition $\mathbf{c}'([L,z]) = [L,c]$ if $[L,c] \in \mathcal{Z}$ and $d_L(c,z) < \varepsilon$. To prove that $\mathbf{c}'$ is well defined, take another point $c' \in L$ satisfying $[L,c'] \in \mathcal{Z}$ and $d_L(c',z) < \varepsilon$, and let us check that $[L,c] = [L,c']$. Choose points $y,y' \in L$ such that $[L,y],[L,y'] \in \mathcal{N}$, $\mathbf{c}([L,y]) = [L,c]$ and $\mathbf{c}([L,y']) = [L,c']$. Then

$$d_L(y,y') \leq d_L(y,c) + d_L(c,z) + d_L(z,c') + d_L(c',y') < 4\varepsilon ,$$

giving $d_L(y,y') \leq \varepsilon$ by Lemma 5.6.2 since $5\varepsilon \leq r$, which implies $[L,c] = [L,c']$ by Lemma 5.9.1. Furthermore $\mathbf{c}'$ is an extension of $\mathbf{c}$ because $d_L(y,\mathcal{C}_{L,y}) < \varepsilon$ for all $[L,y] \in \mathcal{N}$. Note also that $\mathbf{c}'([L,c]) = [L,c]$ for all $[L,c] \in \mathcal{Z}$.

Lemma 5.9.3. *If $[L,z],[L,z'] \in \mathcal{N}'$ and $d_L(z,z') \leq 2\varepsilon$, then $\mathbf{c}'([L,z]) = \mathbf{c}'([L,z'])$.*

Proof. Let $\mathbf{c}'([L,z]) = [L,c]$ and $\mathbf{c}'([L,z']) = [L,c']$. Choose points $[L,y],[L,y'] \in \mathcal{N}$ with $\mathbf{c}([L,y]) = [L,c]$ and $\mathbf{c}([L,y']) = [L,c']$. Then

$$d_L(y,y') \leq d_L(y,c) + d_L(c,z) + d_L(z,z') + d_L(z',c') + d_L(c',y') < 5\varepsilon .$$

From Lemma 5.6.2 and since $5\varepsilon \leq r$, it follows that $[L,c] = [L,c']$. □

Lemma 5.9.4. $\mathbf{c}'$ *is continuous.*

Proof. Take any convergent sequence $[L_i,z_i] \to [L,z]$ in $\mathcal{N}'$. Let $\mathbf{c}'([L_i,z_i]) = [L_i,c_i]$ and $\mathbf{c}'([L,z]) = [L,c]$, and choose points $[L_i,y_i],[L,y] \in \mathcal{N}$ so that $\mathbf{c}([L_i,y_i]) = [L_i,c_i]$ and $\mathbf{c}([L,y]) = [L,c]$. We have

$$d_i(y_i,z_i) \leq d_i(y_i,c_i) + d_i(c_i,z_i) < 2\varepsilon , \quad d_L(y,z) \leq d_L(y,c) + d_L(c,z) < 2\varepsilon .$$

Then, by Lemma 5.6.3, there are points $y_i' \in B_i(z_i,2\varepsilon)$ such that $\lim_i [L_i,y_i'] = [L,y]$ in $\mathcal{M}_*^\infty(n)$ as $i \to \infty$. Thus $[L_i,y_i'] \in \mathcal{N}$ for i large enough, and moreover

$$d_i(y_i,y_i') \leq d_i(y_i,z_i) + d_i(z_i,y_i') < 4\varepsilon ,$$

obtaining $d_i(y_i,y_i') \leq \varepsilon$ by Lemma 5.6.2 since $5\varepsilon \leq r$. By Lemma 5.9.1, it follows that $\mathbf{c}([L_i,y_i']) = \mathbf{c}([L_i,y_i]) = [L_i,c_i]$ for i large enough, giving $\lim_i [L_i,c_i] = [L,c]$ in $\mathcal{M}_*^\infty(n)$ by Lemma 5.9.2. □

We can assume that ε and $\mathcal{N}$ are so small that the following properties hold for all $[L,y] \in \mathcal{N}$ and $z \in B_L(y,\varepsilon)$:

(a) $\exp_L : B_{T_zL}(0_z,\varepsilon) \to B_L(z,\varepsilon)$ is a diffeomorphism; and

(b) $\big\{h \in \operatorname{Iso}(L) \mid h(z) \in \overline{B}(z,4\varepsilon)\big\} = \{\mathrm{id}_L\}$.

Observe that (b) can be assumed by Lemma 5.6.1. Notice also that (a) and (b) hold for all $[L,z] \in \mathcal{Z}$. Let

$$\widehat{\mathcal{N}}' = \{\, [L,\xi] \in \mathcal{T}^{\infty}_{*}(n) \mid \pi([L,\xi]) \in \mathcal{Z},\ |\xi| < \varepsilon \,\} \ .$$

Lemma 5.9.5. $\exp : \widehat{\mathcal{N}}' \to \mathcal{N}'$ *is a homeomorphism.*

Proof. This map is obviously surjective; we will prove that it also injective. For $i \in \{1,2\}$, take points $[L_i,\xi_i] \in \widehat{\mathcal{N}}'$; thus $\xi_i \in T_{c_i}L_i$ for some points $[L_i,c_i] \in \mathcal{Z}$, and we have $\exp([L_i,\xi_i]) = [L_i,z_i]$ for $z_i = \exp_i(\xi_i)$. Suppose that $[L_1,z_1] = [L_2,z_2]$, which means that there is a pointed isometry $\phi : (L_1,z_1) \to (L_2,z_2)$. Then

$$\exp_2 \phi_*(\xi_1) = \phi \exp_1(\xi_1) = \phi(z_1) = z_2 = \exp_2(\xi_2) \ , \tag{5.12}$$

$$d_2(\phi(c_1),c_2) \le d_2(\phi(c_1),z_2) + d_2(z_2,c_2) = d_1(c_1,z_1) + d_2(z_2,c_2) < 2\varepsilon \ . \tag{5.13}$$

We get

$$[L_1,c_1] = \mathbf{c}'([L_1,c_1]) = \mathbf{c}'([L_2,\phi(c_1)]) = [L_2,c_2]$$

by Lemma 5.9.3 and (5.13). So there is an isometry $\psi : L_1 \to L_2$ such that $\psi(c_1) = c_2$. Then the isometry $h = \psi^{-1}\phi : L_1 \to L_1$ satisfies

$$d_1(c_1,h(c_1)) = d_2(c_2,\phi(c_1)) < 2\varepsilon$$

by (5.13), obtaining $h = \mathrm{id}_{L_1}$ by (a). Hence $\phi(c_1) = \psi(c_1) = c_2$, giving $\phi_*(\xi_1) = \xi_2$ by (5.12) and (a) since $\xi_i \in T_{c_i}L_i$. Therefore $\exp : \widehat{\mathcal{N}}' \to \mathcal{N}'$ is bijective.

The continuity of $\exp^{-1} : \mathcal{N}' \to \widehat{\mathcal{N}}'$ is a simple exercise using lemma 5.9.4. □

By Proposition 5.7.11-(i), there is some neighborhood Θ of $[M,x]$ in $\mathcal{M}^{\infty}_{*}(n)$ and some local trivialization $\theta : \pi^{-1}(\Theta) \to \mathbb{R}^n \times \Theta$ of the Riemannian vector bundle $\pi : \mathcal{T}^{\infty}_{*}(n) \to \mathcal{M}^{\infty}_{*}(n)$; in particular, $\theta : \pi^{-1}([L,y]) \to \mathbb{R}^n \times \{[L,y]\} \equiv \mathbb{R}^n$ is a linear isometry for all $[L,y] \in \Theta$. More precisely, according to the proof of Proposition 5.7.11, we can suppose that there is a local section $\sigma : \Theta \to \mathcal{Q}^{\infty}_{*}(n)$ of $\pi : \mathcal{Q}^{\infty}_{*}(n) \to \mathcal{M}^{\infty}_{*}(n)$ so that $\theta([L,\xi]) = (\sigma([L,y])([L,\xi]),[L,y])$ if $\pi_L(\xi) = y$. We can assume that $\mathcal{Z} \subset \Theta$ by Lemma 5.6.4. Hence, by Lemma 5.9.5, the composite

$$\mathcal{N}' \xrightarrow{\exp^{-1}} \widehat{\mathcal{N}}' \xrightarrow{\theta} B^n_{\varepsilon} \times \mathcal{Z}$$

is a homeomorphism $\Phi : \mathcal{N}' \to B^n_{\varepsilon} \times \mathcal{Z}$, where B^n_{ε} denotes the open ball of radius ε centered at the origin in $\mathbb{R}^n$. This shows that $\mathcal{F}_{*,\mathrm{lnp}}(n)$ is a foliated structure of dimension n on $\mathcal{M}^{\infty}_{*,\mathrm{lnp}}(n)$, completing the proof of Theorem 1.3.3-(ii).

Recall that a Riemannian manifold M (or its metric tensor) is called *nowhere locally homogenous* if there is no isometry between distinct open subsets of M. It is easy to see that the proof of [75, Proposition 1] can be adapted to the case of open manifolds, obtaining the following.

Proposition 5.9.6. *For any C^∞ manifold M, the set of nowhere locally homogenous metrics on M is residual in* $\mathrm{Met}(M)$ *with the weak and strong C^∞ topologies.*

Lemma 5.9.7. *There is a nowhere locally homogenous complete Riemannian manifold M such that $\iota(M)$ is dense in $\mathcal{M}^\infty_{*,\mathrm{o}}(n)$.*

Proof. According to the proof of Proposition 5.5.1, there is a countable dense set of points $[M_i, x_i]$ in $\mathcal{M}^\infty_{*,\mathrm{lnp,c}}(n)$ ($i \in \mathbb{N}$). For each i, take some $y_i \in M_i$ satisfying $d_i(x_i, y_i) = \max_{y\in M_i} d_i(x_i, y)$. For all $i \in \mathbb{N}$ and $j, k \in \mathbb{Z}^+$ with $1/j, 1/k < \operatorname{diam} M_i$, let $(M_{ijk}, x_{ijk}, y_{ijk})$ be a copy of (M_i, x_i, y_i), let g_{ijk} be the metric of M_{ijk}, and let Ω_{ijk} be a compact domain in M_{ijk} containing y_{ijk} and with diameter $< 1/j$. Observe that $\widehat{\Omega}_{ijk} := M_{ijk} \smallsetminus \mathrm{Int}(\Omega_{ijk})$ is also a compact domain. Take also corresponding mutually disjoint compact domains Ω'_{ijk} in $\mathbb{R}^n$ so that every bounded subset of $\mathbb{R}^n$ only meets a finite number of them. Let M be the C^∞ connected sum of $\mathbb{R}^n$ with all manifolds M_{ijk} so that the connected sum with each M_{ijk} only involves perturbations inside the interiors of Ω_{ijk} and Ω'_{ijk}. Let g be any Riemannian metric on M whose restriction to each $\widehat{\Omega}_{ijk}$ equals g_{ijk}, and whose restriction to $\mathbb{R}^n \smallsetminus \bigcup_{ijk} \Omega'_{ijk}$ equals the Euclidean metric. Then g is complete and $\iota(M, g)$ is dense in $\mathcal{M}^\infty_*(n)$. With the strong C^∞ topology, $C^\infty(M; TM^* \odot TM^*)$ is a Baire space by [44, Theorem 4.4-(b)]. Since $\mathrm{Met}(M)$ is open in $C^\infty(M; TM^* \odot TM^*)$, and the complete metrics on M form an open subspace $\mathrm{Met}_{\mathrm{com}}(M) \subset \mathrm{Met}(M)$, it follows that $\mathrm{Met}_{\mathrm{com}}(M)$ is a Baire space with the strong C^∞ topology. Hence, by Proposition 5.9.6, there is a nowhere locally homogenous complete metric g' on M so that $\|g - g'\|_{C^k, \widehat{\Omega}_{ijk}, g} < 1/k$ for all i, j and k. Then $\iota(M, g')$ is also dense in $\mathcal{M}^\infty_{*,\mathrm{o}}(n)$. □

By Lemma 5.9.7, $\mathcal{F}_{*,\mathrm{lnp,o}}(n)$ is transitive, showing Theorem 1.3.3-(iii).

Now, for $k \in \{1, 2\}$, let $\Phi_k : \mathcal{N}'_k \to B^n_{\varepsilon_k} \times \mathcal{Z}_k$ be two homeomorphisms constructed as above with maps $\mathbf{c}'_k : \mathcal{N}'_k \to \mathcal{Z}_k$, $\exp : \widehat{\mathcal{N}}'_k \to \mathcal{N}'_k$ and $\sigma_k : \Theta_k \to \mathcal{Q}^\infty_*(n)$.

Lemma 5.9.8. $\Phi_2\Phi_1^{-1} : \Phi_1(\mathcal{N}'_1 \cap \mathcal{N}'_2) \to \Phi_1(\mathcal{N}'_1 \cap \mathcal{N}'_2)$ *is C^∞ (in the sense of Section 4.1).*

Proof. This map has the expression

$$\Phi_2\Phi_1^{-1}(v, [L, c]) = (\Psi(v, [L, c]), \Gamma([L, c])) \ ,$$

where $\Gamma : \mathbf{c}'_1(\mathcal{N}'_1 \cap \mathcal{N}'_2) \to \mathbf{c}'_2(\mathcal{N}'_1 \cap \mathcal{N}'_2)$ is the corresponding holonomy transformation, and $\Psi : \Phi_1(\mathcal{N}'_1 \cap \mathcal{N}'_2) \to \mathbb{R}^n$ is defined by

$$\Psi(v, [L, c]) = \sigma_2([L, c']) \exp^{-1}_{[L,c']} \exp_{[L,c]} \sigma_1([L, c])^{-1}(v) \ ,$$

where $[L, c'] = \Gamma([L, c])$. Let $[L, f] = \sigma_1([L, c])$ and $[L, f'] = \sigma_2([L, c'])$. We can take c' so that $d(c, c') < \varepsilon_1 + \varepsilon_2$, and then

$$\Psi(v, [L, c]) = f' \exp^{-1}_{c'} \exp_c f^{-1}(v) \ .$$

To prove that Ψ is C^∞ in the sense of Section 4.1, fix any $(v,[L,c])\in\Phi_1(\mathcal{N}_1'\cap\mathcal{N}_2')$, and take c', f and f' as above. Let V and $\mathcal{O}$ be open neighborhoods of v and $[L,c]$ in $\mathbb{R}^n$ and $\mathcal{Z}_1$, respectively, such that $\overline{V}\times\mathcal{O}\subset\Phi_1(\mathcal{N}_1'\cap\mathcal{N}_2')$. Take any convergent sequence $[L_i,c_i]\to[L,c]$ in $\mathcal{O}$, and define c_i', f_i and f_i' as before for each i. Notice that $\Psi(v,[L,c])$ and $\Psi(v,[L_i,c_i])$ are defined for all $v\in V$, and let $\psi,\psi_i:V\to\mathbb{R}^n$ be the C^∞ maps given by $\psi(v)=\Psi(v,[L,c])$ and $\psi_i(v)=\Psi(v,[L_i,c_i])$. We have to prove that $\lim_i\psi_i=\psi$ with respect to the weak C^∞ topology.

Let Ω be any compact domain in L such that $\overline{B}_L(c,\varepsilon_1+2\varepsilon_2)\subset\operatorname{Int}(\Omega)$, and thus $\overline{B}_L(c',\varepsilon_2)\subset\operatorname{Int}(\Omega)$ too. Since the sections σ_1 and σ_2 are continuous, there are C^∞ embeddings $\phi_i:\Omega\to L_i$ for i large enough so that $\phi_{i*}(f)=f_i$ and $\lim_i\phi_i^*g_i=g|_\Omega$; in particular, $\phi_i(c)=c_i$. Hence $c_i'\in\phi_i(\operatorname{Int}(\Omega))$ for i large enough, and moreover $\lim_i\phi_{i*}^{-1}(f_i')=f'$ by (b) and Lemma 5.7.12. Observe that $\hat\psi:=\exp_{c'}^{-1}\exp_c$ is defined on $W=f^{-1}(V)\subset B_{T_cL}(0_c,\varepsilon_1)$. It follows that $\hat\psi_i:=\phi_{i*}^{-1}\exp_{c_i'}^{-1}\exp_{c_i}\phi_{i*}$ is also defined on W for i large enough, and moreover $\lim_i\hat\psi_i=\hat\psi$ in the space of C^∞ maps $W\to T_{c'}L$ with the weak C^∞ topology. So

$$\lim_i\phi_{i*}^{-1}(f_i')\,\hat\psi_if^{-1}=f'\hat\psi f^{-1}=\psi$$

in the space of C^∞ maps $V\to\mathbb{R}^n$ with the weak C^∞ topology. Then the result follows because

$$\phi_{i*}^{-1}(f_i')\,\hat\psi_if^{-1}=\phi_{i*}^{-1}(f_i')\,\hat\psi_i\,(\phi_{i*}^{-1}(f_i))^{-1}=f_i'\phi_{i*}\hat\psi_i\phi_{i*}^{-1}f_i^{-1}=f_i'\exp_{c_i'}^{-1}\exp_{c_i}f_i^{-1}=\psi_i\,.\quad\square$$

According to Lemma 5.9.8, $\mathcal{F}_{*,\mathrm{lnp}}(n)$ becomes C^∞ with the above kind of charts. Thus we can consider the tangent bundle $T\mathcal{F}_{*,\mathrm{lnp}}(n)$. For each leaf $\iota(M)$ of $\mathcal{F}_{*,\mathrm{lnp}}(n)$, the canonical homeomorphism $\bar\iota:\operatorname{Iso}(M)\backslash M\to\iota(M)$ is a C^∞ diffeomorphism, and $\iota_{*x}:T_xM\to T_{[M,x]}\mathcal{F}_{*,\mathrm{lnp}}(n)$ is an isomorphism for each $x\in M$. According to Proposition 5.7.11, we get a canonical bijection $T\mathcal{F}_{*,\mathrm{lnp}}(n)\to\mathcal{T}^\infty_{*,\mathrm{lnp}}(n)$ defined by $\iota_{*x}(\xi)\mapsto[M,\xi]$ for $[M,\xi]\in\mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$ and $\xi\in T_xM$. It is an easy exercise to prove that this bijection is an isomorphism of vector bundles. So the Riemannian structure on $\mathcal{T}^\infty_{*,\mathrm{lnp}}(n)$ defined in Proposition 5.7.11 corresponds to a Riemannian structure on $T\mathcal{F}_{*,\mathrm{lnp}}(n)$, which can be easily proved to be C^∞ by using the above kind of flow boxes of $\mathcal{F}_{*,\mathrm{lnp}}(n)$. It is elementary that each isomorphism $\iota_{*x}:T_xM\to T_{[M,x]}\mathcal{F}_{*,\mathrm{lnp}}(n)$ is an isometry. This completes the proof of Theorem 1.3.3-(iv).

Theorem 1.3.3-(v) follows from the following.

Lemma 5.9.9. *The following properties hold for any point $[M,x]\in\mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$, any path $\alpha:I:=[0,1]\to M$ with $\alpha(0)=x$, and any neighborhood $\mathcal{U}$ of $\iota\alpha$ in $C(I,\mathcal{F}_{*,\mathrm{lnp}}(n))$:*

(i) *If* $\alpha(1) = x$ *then, for each* $[N, y] \in \mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$ *close enough to* $[M, x]$*, there is a path* $\beta \in \mathcal{U}$ *with* $\beta(0) = \beta(1) = [N, y]$*.*

(ii) *If* $\alpha(1) \neq x$ *then there is some path* $\beta \in \mathcal{U}$ *with* $\beta(0) \neq \beta(1)$*.*

Proof. Let Ω be a compact domain in M whose interior contains $\alpha(I)$, let $[N, y] \in \mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$, and let $\phi : (\Omega, x) \to (N, y)$ be a pointed C^m embedding satisfying $\|g_M - \phi^* g_N\|_{\Omega, C^m, g_M} < \varepsilon$ for some $m \in \mathbb{Z}^+$ and $\varepsilon > 0$. Let $\beta = \iota\phi\alpha \in C(I, \mathcal{F}_{*,\mathrm{lnp}}(n))$; that is, $\beta(t) = [N, \phi\alpha(t)]$ for each $t \in I$. Observe that $\beta \in \mathcal{U}$ if m and Ω are large enough, and ε is small enough (i.e., if $[N, y]$ is close enough to $[M, x]$). When $\alpha(1) = x$, we get

$$\beta(0) = [N, \phi(x)] = [N, y] = [N, \phi\alpha(1)] = \beta(1) .$$

Suppose now that $\alpha(1) \neq x$. Since $\mathcal{M}^\infty_{*,\mathrm{np}}(n)$ is dense in $\mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$, with the above notation, we can choose $[N, y] \in \mathcal{M}^\infty_{*,\mathrm{np}}(n)$ as close as desired to $[M, x]$. Hence $\iota : N \to \mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$ is injective, giving

$$\beta(0) = \iota\phi(x) \neq \iota\phi\alpha(1) = \beta(1) . \quad \square$$

5.10 Saturated subspaces of $\mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$

Let X be a sequential Riemannian foliated space with complete leaves.

Definition 5.10.1. It is said that X is *covering-determined* when there is a connected pointed covering $(\widetilde{L}_x, \tilde{x})$ of (L_x, x) for all $x \in X$ such that $x_i \to x$ in X if and only if $[\widetilde{L}_{x_i}, \tilde{x}_i]$ is C^∞ convergent to $[\widetilde{L}_x, \tilde{x}]$. When this condition is satisfied with $\widetilde{L}_x = \widetilde{L}_x^{\mathrm{hol}}$ for all $x \in X$, it is said that X is *holonomy-determined*.

Example 5.10.2. (i) The Reeb foliation on S^3 is not covering-determined with any Riemannian metric.

(ii) [55, Example 2.5] is covering-determined but not holonomy-determined.

(iii) $\mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$ is holonomy-determined.

Remark 5.10.1. (i) The condition of being covering-determined is hereditary by saturated subspaces.

(ii) The example X of [55, Example 2.5] can be easily realized as a saturated subspace of a Riemannian foliated space Y where the holonomy coverings of the leaves are isometric to $\mathbb{R}$. Multiplying the leaves by S^1, all holonomy covers of $Y \times S^1$ become isometric to $\mathbb{R} \times S^1$. The metric on $Y \times S^1$ can be modified so that no pair of these holonomy covers are isometric, obtaining a holonomy-determined foliated space, however $X \times S^1$ is not holonomy-determined with any metric. So holonomy-determination is not hereditary by saturated subspaces.

(iii) If X satisfies the covering-determination with the pointed coverings $(\widetilde{L}_x, \tilde{x})$ of (L_x, x) for $x \in X$, then $x = y$ in X if and only if $[\widetilde{L}_x, \tilde{x}] = [\widetilde{L}_y, \tilde{y}]$; in particular, the leaves of X are non-periodic.

(iv) If X is compact and the mapping $x \mapsto [\widetilde{L}_x, \tilde{x}]$ is injective, then the "if" part of Definition 5.10.1 can be deleted.

Proof of Theorem 1.3.4. Any saturated subspace of $\mathcal{M}^{\infty}_{*,\mathrm{lnp}}(n)$ is covering-determined by Example 5.10.2-(iii) and Remark 5.10.1-(i).

Suppose that X satisfies the covering-determination with the pointed covers $(\widetilde{L}_x, \tilde{x})$ of (L_x, x) for $x \in X$. Then the map $\iota : X \to \mathcal{M}^{\infty}_{*,\mathrm{lnp}}(n)$, defined by $\iota(x) = [\widetilde{L}_x, \tilde{x}]$, is a C^∞ foliated embedding whose restrictions to the leaves are isometries. $\square$

Remark 5.10.2. Like in the above proof, a map $\iota^{\mathrm{hol}} : X \to \mathcal{M}^{\infty}_{*}(n)$ is defined by $\iota^{\mathrm{hol}}(x) = [\widetilde{L}^{\mathrm{hol}}_x, \tilde{x}]$, where $\tilde{x} \in \widetilde{L}^{\mathrm{hol}}_x$ is over x. This map may not be continuous [55, Example 2.5], but its restriction to X_0 is continuous by the local Reeb stability theorem, and therefore ι^{hol} is Baire measurable if X is second countable.

Any family $\mathcal{C}$ of complete connected Riemannian n-manifolds defines a closed $\mathcal{F}_*(n)$-saturated subspace $X := \mathrm{Cl}_\infty(\bigcup_{M\in\mathcal{C}} \iota(M)) \subset \mathcal{M}^{\infty}_{*}(n)$. The obvious C^∞ version of arguments of [25] (see also [60, Chapter 10, Sections 3 and 4]) gives the following.

Theorem 5.10.3. *A family $\mathcal{C}$ of complete connected Riemannian n-manifolds is of equibounded geometry if and only if the closed subspace of $\mathcal{M}^{\infty}_{*}(n)$ defined by $\mathcal{C}$ is compact.*

Remark 5.10.3. A version of Theorem 5.10.3 using the Ricci curvature instead of $\mathcal{R}$ can be also proved with the arguments of [10].

For instance, let $\mathcal{M}_*(n, r, C_m) \subset \mathcal{M}_*(n)$ denote the subspace defined by the manifolds of bounded geometry with geometric bound (r, C_m). Each $\mathcal{M}_*(n, r, C_m)$ is compact by Theorem 5.10.3, and the notion of C^∞ convergence in $\mathcal{M}_*(n, r, C_m)$ is equivalent to the convergence in the topology of the Gromov space $\mathcal{M}_*$ [55], [60, Chapter 10]. Nonetheless, this is not the case on the whole of $\mathcal{M}_*(n)$ [17, Section 7.1.4].

Let us study the case of closed subspaces of $\mathcal{M}^{\infty}_{*}(n)$ defined by a single manifold.

Definition 5.10.4. A complete connected Riemannian manifold M is called:

(i) *aperiodic* if, for all $m_i \uparrow \infty$ in $\mathbb{N}$, compact domains $\Omega'_i \subset \Omega_i \subset M$, points $x_i \in \Omega'_i$ and $y_i \in \Omega_i$, and C^{m_i} pointed embeddings $\phi_{ij} : (\Omega_i, x_i) \to (\Omega_j, x_j)$ $(i \leq j)$ and $\psi_i : (\Omega'_i, x_i) \to (\Omega_i, y_i)$ such that

$$\lim_i d(x_i, \partial\Omega'_i)) = \infty\ , \quad \lim_{i,j} \|g - \phi_{ij}^* g\|_{C^{m_i},\Omega_i,g} = \lim_i \|g - \psi_i^* g\|_{C^{m_i},\Omega'_i,g} = 0\ ,$$

we have

$$\lim_i \max\{\, d(x, \psi_i(x)) \mid x \in \Omega'_i \cap \overline{B}(x_i, r)\,\} = 0 \tag{5.14}$$

for some $r > 0$; and

(ii) *weakly aperiodic* if, to get (5.14), besides the conditions of (i), it is also required that there is some $s > 0$ and there are points $z_i \in \Omega'_i$ such that $\phi_{ij}(z_i) = z_j$ and $d(z_i, \psi_i(z_i)) < s$.

Lemma 5.10.5. *The following properties hold for any complete connected Riemannian n-manifold M:*

(i) *M is aperiodic if and only if* $\mathrm{Cl}_\infty(\iota(M)) \subset \mathcal{M}^\infty_{*,\mathrm{np}}(n)$.

(ii) *M is weakly aperiodic if and only if* $\mathrm{Cl}_\infty(\iota(M)) \subset \mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$.

Proof. This is a consequence of Propositions 5.3.2, 5.4.4 and 5.4.5, and using also arguments from the proof of Proposition 5.3.3 for the "if" parts. □

Definition 5.10.6. A complete connected Riemannian manifold M is called *repetitive* if, for every compact domain Ω in M, and all $\varepsilon > 0$ and $m \in \mathbb{N}$, there is a family of C^m embeddings $\phi_i : \Omega \to M$ such that $\bigcup_i \phi_i(\Omega)$ is a net in M and $\|g - \phi_i^* g\|_{C^m,\Omega,g} < \varepsilon$ for all i.

Here, the term *net* in M is used for a subset $A \subset M$ satisfying $\mathrm{Pen}(A, S) = M$ for some $S > 0$.

Lemma 5.10.7. *Let M be a complete connected Riemannian n-manifold of bounded geometry. Then M is repetitive if and only if* $\mathrm{Cl}_\infty(\iota(M))$ *is $\mathcal{F}_*(n)$-minimal.*

Proof. The "only if" part follows easily from Propositions 5.3.2, 5.4.4 and 5.4.5.

To prove the "if" part, assume that $\mathrm{Cl}_\infty(\iota(M))$ is $\mathcal{F}_*(n)$-minimal. Let Ω be a compact domain in M, and take some $m \in \mathbb{N}$ and $\varepsilon > 0$. Take some $x \in M$ and $R > 0$ such that $\Omega \subset B(x, R)$. Let $U = \mathrm{Int}_\infty(D^m_{R,\varepsilon}(M, x))$. Since $\mathrm{Cl}_\infty(\iota(M))$ is compact because M is of bounded geometry (Theorem 5.10.3), there is some $S > 0$ such that $\mathbf{d}_U \leq S$ on $\mathrm{Cl}_\infty(\iota(M))$ by Lemma 5.3.4. Hence $[M, x_i] \in U$ for a net of points x_i in M. Thus there are C^{m+1} pointed local diffeomorphisms $\phi_i\colon (M, x) \rightarrowtail (M, x_i)$ so that $\|g - \phi_i^* g\|_{C^m,\Omega_i,g} < \varepsilon$ for some compact domain $\Omega_i \subset \operatorname{dom}\phi_i$ with $B(x, R) \subset \Omega_i$; in particular, $\Omega \subset \operatorname{dom}\phi_i$ and $\|g - \phi_i^* g\|_{C^m,\Omega,g} < \varepsilon$ for all i, and $\bigcup_i \phi_i(\Omega)$ is a net in M, showing that M is repetitive. □

Proof of Theorem 1.3.5. Suppose that M is non-periodic and has a weakly aperiodic connected covering $\widetilde{M}$. Then $Y = \mathrm{Cl}_\infty(\iota(\widetilde{M}))$ is a compact saturated subspace of $\mathcal{M}^{\infty}_{*,\mathrm{lnp}}(n)$ by Theorem 5.10.3 and Lemma 5.10.5-(ii), and $M \equiv \mathrm{Iso}(\widetilde{M})\backslash\widetilde{M} \xrightarrow{\bar{\iota}} \iota(\widetilde{M})$ is an isometry. Moreover any sequential covering-determined transitive compact Riemannian foliated space can be obtained in this way by Theorem 1.3.4. If $\widetilde{M}$ is also repetitive, then X is minimal by Lemma 5.10.7, completing the proof of (i).

Asume now that M is aperiodic. Then $X = \mathrm{Cl}_\infty(\iota(M))$ is a compact $\mathcal{F}_{*,\mathrm{np}}(n)$-saturated subspace of $\mathcal{M}^{\infty}_{*,\mathrm{np}}(n)$ by Theorem 5.10.3 and Lemma 5.10.5-(i), and moreover $\iota : M \to \iota(M)$ is an isometry. Furthermore the leaves of X have trivial holonomy groups by Theorem 1.3.3-(v). As before, X is minimal if M is also repetitive, showing (ii). $\square$

Chapter 6

Bounded Geometry and Leaves

This chapter contains the proofs of the results about $\widehat{\mathcal{M}}_*^\infty$ stated in Section 1.4.

6.1 (Partial) quasi-equivalences

Let M and N be Riemannian n-manifolds, let $f \in C^\infty(M,\mathbb{E})$ and $h \in C^\infty(N,\mathbb{E})$, and let $x \in M$ and $y \in N$. Recall from Section 1.4 the concepts of an equivalence $(M,f) \to (N,h)$, and a pointed equivalence $(M,f,x) \to (N,h,y)$. Observe that $\|f_*^{(m)}\|_{\Omega^{(m)}}$ makes sense for any n-submanifold $\Omega^{(m)} \subset T^{(m)}M$ because we consider $f_*^{(m)} : T^{(m)}M \to T^{(m)}\mathbb{E} \equiv \mathbb{E}^{2^m}$, with values in a separable Hilbert space. Note also that $(\phi^*h)_*^{(m)} = h_*^{(m)} \circ \phi_*^{(m)}$ for any C^m map $\phi : M \to N$.

Definition 6.1.1. Let $\lambda \geq 1$ and $\varepsilon \geq 0$, and let $\phi : M \to N$ be a C^1 map. It is said that $\phi : (M,f) \to (N,h)$ is a $((\lambda,\varepsilon)$-$)$ *quasi-equivalence of order* $m \in \mathbb{N}$ if it is C^{m+1}, $\phi_*^{(m)} : T^{(m),\leq 1}M \to T^{(m)}N$ is a (λ-) quasi-isometry, and $\|f_*^{(m)} - (\phi^*h)_*^{(m)}\|_{T^{(m)}M} \leq \varepsilon$. If moreover distinguished points x and y are preserved, then $\phi : (M,f,x) \to (N,h,y)$ is called a *pointed quasi-equivalence of order* m. If there is a quasi-equivalence $(M,f) \to (N,h)$ (respectively, $(M,f,x) \to (N,h,y)$), then (M,f) and (N,h) (respectively, (M,f,x) and (N,h,y)) are called *quasi-equivalent*.

Remark 6.1.1. (i) Any (λ,ε)-quasi-equivalence of order $m \geq 1$ is also a (λ,ε)-quasi-equivalence of order $m-1$.

(ii) For integers $0 \leq m' \leq m$, if ϕ is a (λ,ε)-quasi-equivalence of order m, then $\phi_*^{(m')}$ is a (λ,ε)-quasi-equivalence of order $m-m'$.

For a submanifold $\Omega \subset M$ and $f \in C^\infty(M,\mathbb{E})$, the notation (Ω,f) is used for $(\Omega,f|_\Omega)$.

Proposition 6.1.2. *The following properties hold for any $m \in \mathbb{N}$, $\lambda,\mu \geq 1$ and $\varepsilon,\delta \geq 0$:*

(i) *There is some $\nu \geq 1$, depending on m, λ and μ, such that, if $\phi : (M, f) \to (N, h)$ is a (λ, ε)-quasi-equivalence and $\psi : (N, h) \to (L, u)$ a (μ, δ)-quasi-equivalence, both of them of order m, then $\psi \circ \phi : (M, f) \to (L, u)$ is a $(\nu, \varepsilon + \delta)$-quasi-equivalence of order m.*

(ii) *There are some $\nu' \geq 1$, depending on m and λ, such that, if $\phi : (M, f) \to (N, h)$ is a (λ, ε)-quasi-equivalence of order m and a diffeomorphism, then $\phi^{-1} : (N, h) \to (M, f)$ is a (ν', ε)-quasi-equivalence of order m.*

Proof. By Proposition 5.1.9, we only have to check the conditions on the $\mathbb{E}$-valued functions. Thus (i) follows because, for each $\xi \in T^{(m)}M$, we have

$$\begin{aligned}&\left\|f_*^{(m)}(\xi) - ((\psi \circ \phi)^* u)_*^{(m)}(\xi)\right\| \\ &\qquad \leq \left\|f_*^{(m)}(\xi) - (\phi^* h)_*^{(m)}(\xi)\right\| + \left\|h_*^{(m)}\left(\phi_*^{(m)}(\xi)\right) - (\psi^* u)_*^{(m)}\left(\phi_*^{(m)}(\xi)\right)\right\| \leq \varepsilon + \delta\,.\end{aligned}$$

Similarly, (ii) follows because, for each $\zeta \in T^{(m)}N$,

$$\left\|h_*^{(m)}(\zeta) - ((\phi^{-1})^* f)_*^{(m)}(\zeta)\right\| = \left\|(\phi^* h)_*^{(m)}\left((\phi^{-1})_*^{(m)}(\zeta)\right) - f_*^{(m)}\left((\phi^{-1})_*^{(m)}(\zeta)\right)\right\| \leq \varepsilon\,. \quad \square$$

Corollary 6.1.3. *"Being quasi-equivalent with order m" is an equivalence relation on the sets of pairs (M, f) and triples (M, f, x).*

Now, suppose that M and N are connected, complete and without boundary.

Definition 6.1.4. Fix $m \in \mathbb{N}$, $R > 0$, $\lambda \geq 1$ and $\varepsilon \geq 0$. Let $\phi\colon (M, x) \rightarrowtail (N, y)$ be a C^{m+1} pointed local diffeomorphism, and let $f \in C^\infty(M, \mathbb{E})$ and $h \in C^\infty(N, \mathbb{E})$. It is said that $\phi\colon (M, f, x) \rightarrowtail (N, h, y)$ is an $(m, R, \lambda, \varepsilon)$-*pointed local quasi-equivalence*, or a *local quasi-equivalence* of *type* $(m, R, \lambda, \varepsilon)$, if there is some compact domain $\Omega^{(m)} \subset \operatorname{dom} \phi_*^{(m)}$ such that $B_M^{(m)}(x, R) \subset \Omega^{(m)}$ and $\phi_*^{(m)} : (\Omega^{(m)}, f_*^{(m)}) \to (T^{(m)}N, h_*^{(m)})$ is a (λ, ε)-quasi-equivalence.

Remark 6.1.2. (i) Any $(m, R, \lambda, \varepsilon)$-pointed local quasi-equivalence is also a pointed local quasi-equivalence of type $(m', R', \lambda', \varepsilon')$ for $0 \leq m' \leq m$, $0 < R' < R$, $\lambda' > \lambda$ and $\varepsilon' > \varepsilon$.

(ii) Consider integers $0 \leq m' \leq m$, any pointed C^{m+1} local diffeomorphism $\phi : (M, x) \rightarrowtail (N, y)$, and any $f \in C^\infty(M, \mathbb{E})$ and $h \in C^\infty(N, \mathbb{E})$. Then $\phi : (M, f, x) \rightarrowtail (N, h, y)$ is a pointed local quasi-equivalence of type $(m, R, \lambda, \varepsilon)$ if and only if

$$\phi_*^{(m')} : (T^{(m')}M, f_*^{(m')}, x) \rightarrowtail (T^{(m')}N, h_*^{(m')}, y)$$

is a pointed local quasi-equivalence of type $(m - m', R, \lambda, \varepsilon)$.

(iii) If there is an $(m,R,\lambda,\varepsilon)$-pointed local quasi-equivalence $(M,f,x) \rightarrowtail (N,h,y)$, then, for all $R' < R$, $\lambda' > \lambda$ and $\varepsilon' > \varepsilon$, there is a C^∞ $(m,R',\lambda',\varepsilon')$-pointed local quasi-equivalence $(M,f,x) \rightarrowtail (N,h,y)$ by [44, Theorem 2.7].

Lemma 6.1.5. *The following properties hold:*

(i) *Suppose that $\phi : (M,f,x) \rightarrowtail (N,h,y)$ and $\psi : (N,h,y) \rightarrowtail (L,u,z)$ are pointed local quasi-equivalences of types $(m,R,\lambda,\varepsilon)$ and $(m,\lambda R,\lambda',\varepsilon')$, respectively. Then $\psi\circ\phi : (M,f,x) \rightarrowtail (L,u,z)$ is an $(m,R,\lambda\lambda',\varepsilon+\varepsilon')$-pointed local quasi-equivalence.*

(ii) *If $\phi : (M,f,x) \rightarrowtail (N,h,y)$ is an $(m,\lambda R,\lambda,\varepsilon)$-pointed local quasi-isometry, then $\phi^{-1} : (N,h,y) \rightarrowtail (M,f,x)$ is an $(m,R,\lambda,\varepsilon)$-pointed local quasi-isometry.*

Proof. To prove (i), take compact domains, $\Omega^{(m)} \subset T^{(m)}M$ and $\Omega'^{(m)} \subset T^{(m)}N$, such that $B_M^{(m)}(x,R) \subset \Omega^{(m)}$, $B_N^{(m)}(x,\lambda R) \subset \Omega'^{(m)}$, $\phi_*^{(m)} : (\Omega^{(m)}, f_*^{(m)}) \to (T^{(m)}N, h_*^{(m)})$ is a (λ,ε)-quasi-equivalence, and $\psi_*^{(m)} : (\Omega'^{(m)}, h_*^{(m)}) \to (T^{(m)}L, u_*^{(m)})$ is a (λ',ε')-quasi-equivalence. According to the proof of Lemma 5.2.3 (i), there is a compact domain $\Omega_0^{(m)} \subset T^{(m)}M$ such that $B_M^{(m)}(x,R) \subset \Omega_0^{(m)}$ and $\phi_*^{(m)}(\Omega_0^{(m)}) \subset \Omega'^{(m)}$. Then $(\psi\circ\phi)_*^{(m)} : \Omega_0^{(m)} \to T^{(m)}L$ is a $\lambda\lambda'$-quasi-isometry by Remark 5.1.1 (v). Moreover, for each $\xi \in \Omega_0^{(m)}$,

$$\begin{aligned}\left\|f_*^{(m)}(\xi) - ((\psi\circ\phi)^*u)_*^{(m)}(\xi)\right\| \\ \le \left\|f_*^{(m)}(\xi) - (\phi^*h)_*^{(m)}(\xi)\right\| + \left\|h_*^{(m)}\left(\phi_*^{(m)}(\xi)\right) - (\psi^*u)_*^{(m)}\left(\phi_*^{(m)}(\xi)\right)\right\| \le \varepsilon+\varepsilon' .\end{aligned}$$

So $\psi\circ\phi : (M,f,x) \rightarrowtail (L,u,z)$ is an $(m,R,\lambda\lambda',\varepsilon+\varepsilon')$-pointed local quasi-equivalence.

To prove (ii), let $\Omega^{(m)} \subset T^{(m)}M$ be a compact domain such that $B_M^{(m)}(x,R) \subset \Omega^{(m)}$, and $\phi_*^{(m)} : (\Omega^{(m)}, f_*^{(m)}) \to (T^{(m)}N, h_*^{(m)})$ is a (λ,ε)-quasi-equivalence. According to the proof of Lemma 5.2.3 (ii), the compact domain $\Omega'^{(m)} := \phi_*^{(m)}(\Omega^{(m)}) \subset T^{(m)}N$ contains $B_N^{(m)}(y,R)$. Then $(\phi^{-1})_*^{(m)} = (\phi_*^{(m)})^{-1} : \Omega'^{(m)} \to T^{(m)}M$ is a λ-quasi-isometry by Remark 5.1.1 (vi). Moreover, for each $\xi \in \Omega'^{(m)}$,

$$\left\|h_*^{(m)}(\xi) - ((\phi^{-1})^*f)_*^{(m)}(\xi)\right\| \le \left\|(\phi^*h)_*^{(m)}\left((\phi^{-1})_*^{(m)}(\xi)\right) - f_*^{(m)}\left((\phi^{-1})_*^{(m)}(\xi)\right)\right\| \le \varepsilon .$$

So $\phi^{-1} : (N,h,y) \rightarrowtail (M,f,x)$ is an $(m,R,\lambda,\varepsilon)$-pointed local quasi-equivalence. □

6.2 The C^∞ topology on $\widehat{\mathcal{M}}_*(n)$

Definition 6.2.1. For $m \in \mathbb{N}$, $R, r > 0$, let $\widehat{U}_{R,r}^m$ be the set of pairs $([M,f,x],[N,h,y]) \in \widehat{\mathcal{M}}_*(n) \times \widehat{\mathcal{M}}_*(n)$ satisfying that there is some $(m,R,\lambda,\varepsilon)$-pointed local quasi-equivalence $(M,f,x) \rightarrowtail (N,h,y)$ for some $\lambda \in [1,e^r)$ and $\varepsilon \in (0,r)$.

Proposition 6.2.2. *The following properties*[1] *hold for all $m, m' \in \mathbb{N}$ and $R, S, r, s > 0$:*

(i) $(\widehat{U}^m_{e^r R,r})^{-1} \subset \widehat{U}^m_{R,r}$.

(ii) $\widehat{U}^{m_0}_{R_0,r_0} \subset \widehat{U}^m_{R,r} \cap \widehat{U}^{m'}_{S,s}$, *where* $m_0 = \max\{m, m'\}$, $R_0 = \max\{R, S\}$ *and* $r_0 = \min\{r, s\}$.

(iii) $\Delta \subset \widehat{U}^m_{R,r}$.

(iv) $\widehat{U}^m_{R,r} \circ \widehat{U}^m_{e^r R,s} \subset \widehat{U}^m_{R,r+s}$.

Proof. Properties (ii) and (iii) are elementary, while Properties (i) and (iv) are consequences of Lemma 6.1.5. □

Proposition 6.2.3. $\bigcap_{R,r>0} \widehat{U}^m_{R,r} = \Delta$ *for all $m \in \mathbb{N}$.*

Proof. We only have to prove "$\subset$" by Proposition 6.2.2-(iii). For $([M, f, x], [N, h, y]) \in \bigcap_{R,r>0} \widehat{U}^m_{R,r}$, there is a sequence of pointed local quasi-equivalences $\phi_i : (M, f, x) \rightarrowtail (N, h, y)$, with corresponding types $(m, R_i, \lambda_i, \varepsilon_i)$, such that $R_i \uparrow \infty$, $\lambda_i \downarrow 1$ and $\varepsilon_i \downarrow 0$ as $i \to \infty$. According to the proof of Proposition 5.3.3, for each i, there is some subsequence $\phi_{k(i,l)}$ whose restriction to $B_M(x, R_i)$ converges to some pointed isometric immersion $\psi_i : (B_M(x, R_i), x) \to (N, y)$ in the weak C^m topology, $\psi_{i+1}|_{B_M(x,R_i)} = \psi_i$ for all i, and the combination of the maps ψ_i is a pointed isometry $\psi : (M, x) \to (N, y)$. For every $x' \in M$ and $\varepsilon > 0$, there are some i and $\delta > 0$ so that $x' \in B_M(x, R_i)$, $\varepsilon_i \le \varepsilon/2$, and $\|h(y') - h(y'')\| < \varepsilon/2$ if $d_N(y', y'') < \delta$ for all $y', y'' \in \overline{B}_M(x, R_i)$. Moreover there is some l such that $d_N(\phi_{k(i,l)}(x'), \psi_i(x')) < \delta$. Hence

$$\|f(x') - h \circ \psi(x')\| \le \|f(x') - h \circ \phi_{k(i,l)}(x')\| + \|h \circ \phi_{k(i,l)}(x') - h \circ \psi(x')\| < \varepsilon_i + \varepsilon/2 \le \varepsilon.$$

Since x' and ε are arbitrary, it follows that $\psi : (M, f, x) \to (N, h, y)$ is an equivalence, and therefore $[M, f, x] = [N, h, y]$. □

By Propositions 6.2.2 and 6.2.3, the sets $\widehat{U}^m_{R,r}$ form a base of entourages of a separating uniformity on $\widehat{\mathcal{M}}_*(n)$, which is called the C^∞ *uniformity*.

Definition 6.2.4. For $R, r > 0$, $m \in \mathbb{N}$, let $\widehat{D}^m_{R,r}$ be the set of pairs $([M, f, x], [N, h, y]) \in \widehat{\mathcal{M}}_*(n) \times \widehat{\mathcal{M}}_*(n)$ such that there is some C^{m+1} pointed local diffeomorphism $\phi\colon (M, x) \rightarrowtail (N, y)$ so that $\|g_M - \phi^* g_N\|_{C^m,\Omega,g_M} < r$ and $\|f - \phi^* h\|_{C^m,\Omega,g_M} < r$ for some compact domain $\Omega \subset \operatorname{dom}\phi$ with $B_M(x, R) \subset \Omega$.

[1] The following standard notation is used for a set X and relations $U, V \subset X \times X$:

$$U^{-1} = \{\, (y, x) \in X \times X \mid (x, y) \in U \,\},$$
$$V \circ U = \{\, (x, z) \in X \times X \mid \exists y \in X \text{ so that } (x, y) \in U \text{ and } (y, z) \in V \,\}.$$

Moreover the diagonal of $X \times X$ is denoted by Δ.

Remark 6.2.1. By (4.3), and its version for $\mathbb{E}$-valued functions, a sequence $[M_i, f_i, x_i] \in \widehat{\mathcal{M}}_*(n)$ is C^∞ convergent to $[M, f, x] \in \widehat{\mathcal{M}}_*(n)$ if and only if it is eventually in[2] $\widehat{D}^m_{R,r}(M, f, x)$ for arbitrary $m \in \mathbb{N}$ and $R, r > 0$.

Proposition 6.2.5. *The following properties hold:*

(i) *For all $R, r > 0$, if $0 < r' \leq \min\{1 - e^{-2r}, e^{2r} - 1, r\}$, then $\widehat{D}^0_{R,r'} \subset \widehat{U}^0_{R,r}$.*

(ii) *For all $m \in \mathbb{Z}^+$, $R, r > 0$ and $[M, f, x] \in \widehat{\mathcal{M}}_*(n)$, there is some $r' > 0$ such that $\widehat{D}^m_{R,r'}(M, f, x) \subset \widehat{U}^m_{R,r}(M, f, x)$.*

Proof. Let us show (i). If $([M, f, x], [N, h, y]) \in \widehat{D}^0_{R,r'}$, then there is a C^1 pointed local diffeomorphism $\phi : (M, x) \rightarrowtail (N, y)$ such that $r'_0 := \|g_M - \phi^* g_N\|_{C^0,\Omega,g_M} < r'$ and $\varepsilon_0 := \|f - \phi^* h\|_{C^0,\Omega,g_M} < r'$ for some compact domain $\Omega \subset \operatorname{dom}\phi$ with $B_M(x, R) \subset \Omega$. Take some $\lambda \in [1, e^r)$ such that $r'_0 \leq \min\{1 - \lambda^{-2}, \lambda^2 - 1\}$. According to the proof of Proposition 5.4.4 (i), $\phi : \Omega \to N$ is a λ-quasi-isometry. Since moreover $\|f - \phi^* h\|_\Omega \leq \varepsilon_0$, it follows that ϕ is a $(0, R, \lambda, r'_0, \varepsilon_0)$-pointed local quasi-equivalence, obtaining that $([M, f, x], [N, h, y]) \in \widehat{U}^0_{R,r}$.

Let us prove (ii). Take $m \in \mathbb{Z}^+$, $R, r > 0$ and $[M, f, x] \in \widehat{\mathcal{M}}_*(n)$. Let $\mathcal{U}$ be a finite collection of charts of M with domains U_a, and let $\mathcal{K} = \{K_a\}$ be a family of compact subsets of M, with the same index set as $\mathcal{U}$, such that $K_a \subset U_a$ for all a, and $\overline{B}_M(x, R) \subset \operatorname{Int}(K)$ for $K = \bigcup_a K_a$. Let $r' > 0$, to be fixed later. For any $[N, h, y] \in \widehat{D}^m_{R,r'}(M, x)$, there is a C^{m+1} pointed local diffeomorphism $\phi\colon (M, x) \rightarrowtail (N, y)$ so that $\|g_M - \phi^* g_N\|_{C^m,\Omega,g_M} < r'$ and $\|f - \phi^* h\|_{C^m,\Omega,g_M} < r'$ for some compact domain $\Omega \subset \operatorname{dom}\phi \cap \operatorname{Int}(K)$ with $B_M(x, R) \subset \Omega$. By continuity, there is another compact domain $\Omega' \subset \operatorname{dom}\phi \cap \operatorname{Int}(K)$ such that $\Omega \subset \operatorname{Int}(\Omega')$, $\|g_M - \phi^* g_N\|_{C^m,\Omega',g_M} < r'$ and $\|f - \phi^* h\|_{C^m,\Omega',g_M} < r'$. According to the proof of Proposition 5.4.4 (i), if r' is small enough (depending on m, R, r and $[M, x]$), then there is some compact domain $\Omega^{(m)} \subset T^{(m)}M$ such that $B^{(m)}_M(x, R) \subset \Omega^{(m)} \subset \pi^{-1}(\Omega')$, where $\pi : T^{(m)}M \to M$, and $\phi^{(m)}_* : \Omega^{(m)} \to T^{(m)}N$ is a λ-quasi-isometry for some $\lambda \in [1, e^r)$. Given $\varepsilon \in (0, r)$, choose some $C \geq 1$ satisfying (4.3) for $\mathbb{E}$-valued functions with $\mathcal{U}$, $\mathcal{K}$, Ω' and g, and, according to Lemma 4.2.2-(i), choose some $\varepsilon' > 0$ such that

$$\|f - \phi^* h\|_{C^m,\Omega',\mathcal{U},\mathcal{K}} < \varepsilon' \implies \|f^{(m)}_* - (\phi^* h)^{(m)}_*\|_{\Omega^{(m)}} < \varepsilon\ .$$

Suppose that $r' \leq \varepsilon'/C$. Then

$$\begin{aligned} \|f - \phi^* h\|_{C^m,\Omega',g_M} < r' &\implies \|f - \phi^* h\|_{C^m,\Omega',\mathcal{U},\mathcal{K}} < Cr' \leq \varepsilon' \\ &\implies \|f^{(m)}_* - (\phi^* h)^{(m)}_*\|_{\Omega^{(m)}} < \varepsilon\ . \end{aligned}$$

[2]Given a set X, for $U \subset X \times X$ and $x \in X$, let $U(x) = \{\, y \in Y \mid (x, y) \in U \,\}$. In the case of $U \subset \widehat{\mathcal{M}}_*(n) \times \widehat{\mathcal{M}}_*(n)$ and $[M, f, x] \in \widehat{\mathcal{M}}_*(n)$, we simply write $U(M, f, x)$.

Hence ϕ is an $(m,R,\lambda,\varepsilon)$-pointed local quasi-equivalence $(M,f,x) \rightarrowtail (N,h,y)$, and therefore $[N,h,y] \in \widehat{U}^{(m)}_{R,r}(M,f,x)$. $\square$

Proposition 6.2.6. *The following properties hold:*

(i) *For all $R,r>0$, if $e^{2r'}-e^{-2r'} \le r$, then $\widehat{U}^0_{R,r'} \subset \widehat{D}^0_{R,r}$.*

(ii) *For all $m \in \mathbb{Z}^+$, $R,r>0$ and $[M,f,x] \in \widehat{\mathcal{M}}_*(n)$, there is some $r'>0$ such that $\widehat{U}^m_{R,r'}(M,f,x) \subset \widehat{D}^m_{R,r}(M,f,x)$.*

Proof. Let us show (i). If $([M,f,x],[N,h,y]) \in \widehat{U}^0_{R,r'}$, then there is a $(0,R,\lambda,\varepsilon)$-pointed local quasi-equivalence $\phi : (M,f,x) \rightarrowtail (N,h,y)$ for some $\lambda \in [1,e^{r'})$ and $\varepsilon \in (0,r')$. Thus there is some compact domain $\Omega \subset \operatorname{dom}\phi$ such that $B_M(x,R) \subset \Omega$ and $\phi : (\Omega,f) \to (N,h)$ is a (λ,ε)-quasi-equivalence. According to the proof of Proposition 5.4.5 (i), $\|g_M - \phi^* g_N\|_{C^0,\Omega,g} < r$. So $([M,f,x],[N,h,y]) \in \widehat{D}^0_{R,r}$.

Let us prove (ii). Let $m \in \mathbb{Z}^+$, $R,r>0$ and $[M,f,x] \in \widehat{\mathcal{M}}_*(n)$. Take $\mathcal{U}$, $\mathcal{K}$ and K like in the proof of Proposition 6.2.5-(ii). Let $r'>0$, to be fixed later. For any $[N,h,y] \in \widehat{U}^m_{R,r'}(M,x)$, there is an $(m,R,\lambda,\varepsilon)$-pointed local quasi-equivalence $\phi : (M,f,x) \rightarrowtail (N,h,y)$ for some $\lambda \in [1,e^{r'})$ and $\varepsilon \in (0,r')$. Thus there is a compact domain $\Omega^{(m)} \subset \operatorname{dom}\phi^{(m)}_* \cap \operatorname{Int}(K^{(m)})$ so that $B^{(m)}_M(x,R) \subset \Omega^{(m)}$ and $\phi^{(m)}_* : (\Omega^{(m)}, f^{(m)}_*) \to (T^{(m)}N, h^{(m)}_*)$ is a (λ,ε)-quasi-equivalence. According to the proof of Proposition 5.4.5 (ii), there are compact domains, $\Omega'^{(m)} \subset \operatorname{dom}\phi^{(m)}_*$ and $\Omega \subset M$, such that $\Omega^{(m)} \subset \operatorname{Int}(\Omega'^{(m)})$, $\Omega^{(m)} \cap M \subset \Omega \subset \operatorname{Int}(\Omega'^{(m)})$, and $\|g_M - \phi^* g_N\|_{C^m,\Omega,g} < r$ if r' is small enough; in particular, $B_M(x,R) \subset \Omega$ because M is a totally geodesic Riemannian submanifold of $T^{(m)}M$. Take some $C \ge 1$ satisfying (4.3) for $\mathbb{E}$-valued functions with $\mathcal{U}$, $\mathcal{K}$, Ω and g_M. With the notation of Section 4.2, for $\rho>0$ and $n+1 \le \mu \le 2^m n$, let $\sigma^{(m)}_{a,\rho,\mu} : U_a \to U^{(m)}_a$ be the section of each $\pi : U^{(m)}_a \to U_a$ of the type used in Lemma 4.2.2-(ii). Since $\Omega \subset \operatorname{Int}(\Omega'^{(m)})$, there is some $\rho>0$ so that $\sigma^{(m)}_{\rho,\mu}(K_a \cap \Omega) \subset \Omega'^{(m)}$ for all a and μ. Thus, by Lemma 4.2.2-(ii), there is some $\varepsilon'>0$, depending on r and ρ, such that

$$\|f^{(m)}_* - (\phi^* h)^{(m)}_*\|_{\Omega'^{(m)}} < \varepsilon' \implies \|f^* - \phi^* h\|_{C^m,\Omega,\mathcal{U},\mathcal{K}} < r/C\ .$$

Suppose that moreover $r' < \varepsilon'$, and therefore $\varepsilon < \varepsilon'$. Then

$$\begin{aligned}\|f^{(m)}_* - (\phi^* h)^{(m)}_*\|_{\Omega'^{(m)}} \le \varepsilon < \varepsilon' &\implies \|f - \phi^* h\|_{C^m,\Omega,\mathcal{U},\mathcal{K}} < r/C \\ &\implies \|f - \phi^* h\|_{C^m,\Omega,g} < r\ ,\end{aligned}$$

showing that $[N,h,y] \in \widehat{D}^{(m)}_{R,r}(M,f,x)$. $\square$

Corollary 6.2.7. *The C^∞ convergence in $\widehat{\mathcal{M}}_*(n)$ describes the topology induced by the C^∞ uniformity.*

Proof. This is a direct consequence of Remark 6.2.1 and Propositions 6.2.5 and 6.2.6. □

According to Corollary 6.2.7, the C^∞ uniformity induces what was called the C^∞ topology in Section 1.4. Recall that the corresponding space is denoted by $\widehat{\mathcal{M}}_*^\infty(n)$, and the notation $\widehat{\mathrm{Cl}}_\infty$ is used for the closure operator in $\widehat{\mathcal{M}}_*^\infty(n)$.

Proposition 6.2.8. *$\widehat{\mathcal{M}}_*^\infty(n)$ is separable.*

Proof. According to the proof of Proposition 5.5.1, there is a countable family $\mathcal{C}$ of C^∞ compact manifolds containing exactly one representative of every diffeomorphism class, and, for every $M \in \mathcal{C}$, there is a countable dense subset $\mathcal{G}_M$ of the space of metrics on M with the C^∞ topology. Take also countable dense subsets, $\mathcal{D}_M \subset M$ and $\mathcal{F}_M \subset C^\infty(M,\mathbb{E})$. Then, like in the proof of Proposition 5.5.1, the countable set

$$\{\,[(M,g),f,x] \mid M \in \mathcal{C},\ g \in \mathcal{G}_M,\ x \in \mathcal{D}_M,\ f \in \mathcal{F}_M\,\} \tag{6.1}$$

is dense in $\widehat{\mathcal{M}}_*^\infty(n)$. □

Proposition 6.2.9. *The C^∞ uniformity is complete and metrizable.*

Proof. According to [77, Corollary 38.4], the C^∞ uniformity on $\widehat{\mathcal{M}}_*(n)$ is metrizable because it is separating and the sets $\widehat{U}_{k,1/k}$ ($k \in \mathbb{Z}^+$) form a countable base of entourages. To check that this uniformity is complete, consider an arbitrary Cauchy sequence $[M_i,f_i,x_i]$ in $\widehat{\mathcal{M}}_*(n)$. We have to prove that $[M_i,f_i,x_i]$ is convergent in $\widehat{\mathcal{M}}_*^\infty(n)$. By taking a subsequence if necessary, we can suppose that $([M_i,f_i,x_i],[M_{i+1},x_{i+1},f_{i+1}]) \in U_{R_i,r_i}^{m_i}$ for sequences, $m_i \uparrow \infty$ in $\mathbb{N}$, and $R_i \uparrow \infty$ and $r_i \downarrow 0$ in $\mathbb{R}^+$, such that $\sum_i r_i < \infty$, and $R_{i+1} \geq e^{r_i}R_i$ for all i. Let $\bar{r}_i = \sum_{j\geq i} r_j$. Consider other sequences $R_i', R_i'' \uparrow \infty$ in $\mathbb{R}^+$ such that $R_i' < R_i'' \leq e^{-\bar{r}_i}R_i$ and $R_{i+1}' \geq e^{r_i}R_i''$.

For each i, there is some $(m_i,R_i,\lambda_i,\varepsilon_i)$-pointed local quasi-equivalence $\phi_i\colon (M_i,x_i) \rightarrowtail (M_{i+1},x_{i+1})$, for some $\lambda_i \in (1,e^{r_i})$ and $\varepsilon_i \in (0,r_i)$. Then $\bar{\lambda}_i := \prod_{j\geq i}\lambda_j < e^{\bar{r}_i}$ and $\bar{\varepsilon}_i := \sum_{j\geq i}\varepsilon_j < \bar{r}_i$. Moreover each ϕ_i can be assumed to be C^∞ by Remark 6.1.2-(iii). For $i<j$, the pointed local quasi-equivalence $\psi_{ij} = \phi_{j-1}\circ\cdots\circ\phi_i : (M_i,f_i,x_i) \rightarrowtail (M_j,x_j,f_j)$ is of type $(m_i,R_i/\bar{\lambda}_i,\bar{\lambda}_i,\bar{r}_i)$ by Lemma 6.1.5-(i).

For $i,m \in \mathbb{N}$, let

$$\begin{gathered} B_i = B_i(x_i,R_i)\;, \qquad B_i' = B_i(x_i,R_i')\;, \qquad B_i'' = B_i(x_i,R_i'')\;,\\ B_i^{(m)} = B_i^{(m)}(x_i,R_i)\;, \quad B_i'^{(m)} = B_i^{(m)}(x_i,R_i')\;, \quad B_i''^{(m)} = B_i^{(m_i)}(x_i,R_i'')\;. \end{gathered}$$

A bar is added to this notation when the corresponding closed balls are considered. We have $\phi_i(\overline{B}_i) \subset B_{i+1}$ because $R_{i+1} > \lambda_i R_i$, and $\phi_{i*}^{(m_i)}(\overline{B}_i''^{(m_i)}) \subset B_{i+1}'^{(m_i)} \subset B_{i+1}'^{(m_{i+1})}$ since $R_{i+1}' > \lambda_i R_i''$ and $g_{i+1}^{(m_i)}$ is the restriction of $g_{i+1}^{(m_i+1)}$. Furthermore $B_i'' \subset \operatorname{dom}\psi_{ij}$

and $B_i''^{(m_i)} \subset \operatorname{dom}\psi_{ij*}^{(m_i)}$ for $i < j$ because $R'' \le R_i/\bar\lambda_i$. Therefore $\psi_{ij}(B_i) \subset B_j$ and $\psi_{ij*}^{(m_i)}(B_i''^{(m_i)}) \subset B_j'^{(m_j)}$. Take compact domains, $\Omega_i \subset M_i$ and $\Omega_i^{(m_i)} \subset T^{(m_i)}M_i$, such that $B_i' \subset \Omega_i \subset \operatorname{Int}(\Omega_i^{(m_i)})$ and $B_i'^{(m_i)} \subset \Omega_i^{(m_i)} \subset B_i''^{(m_i)}$; thus $\Omega_i \subset B_i''$ since M_i is a totally geodesic Riemannian submanifold of $T^{(m_i)}M_i$.

According to the proof of Proposition 5.5.2, there is a pointed complete connected Riemannian manifold $(\widehat{M},\hat x)$, and, for each i, there is some C^∞ pointed map $\psi_i : (B_i, x_i) \to (\widehat{M},\hat x)$ such that $\psi_{i*}^{(m_i)} : \Omega_i^{(m_i)} \to T^{(m_i)}\widehat{M}$ is a $\bar\lambda_i$-quasi-isometry, and $\psi_i = \psi_j \circ \psi_{ij}$ for $j \ge i$. Let $\widehat{B}_i = \psi_i(B_i)$, $\widehat{\Omega}_i = \psi_i(\Omega_i)$ and $\widehat{\Omega}_i^{(m_i)} = \psi_{i*}^{(m_i)}(\Omega_i^{(m_i)})$. Let $\hat f_i \in C^\infty(\widehat{B}_i,\mathbb{E})$ be determined by $\psi_i^*\hat f_i = f_i|_{B_i}$.

Claim 6.2.1. For all i, the sequence $\hat f_j|_{\widehat\Omega_i}$ $(j \ge i)$ is convergent in $C^{m_i}(\widehat\Omega_i,\mathbb{E})$.

This assertion follows by showing that the restrictions of the functions $f_{ij} := \psi_{ij}^* f_j$ to Ω_i, for $j \ge i$, form a convergent sequence in $C^{m_i}(\Omega_i,\mathbb{E})$. Equivalently, we show that $f_{ij}|_{\Omega_i}$ is a Cauchy sequence with respect to $\|\ \|_{C^{m_i},\Omega_i,g_i}$. For $k \ge j \ge i$,

$$\begin{aligned}
\left\|f_{ij*}^{(m_i)} - f_{ik*}^{(m_i)}\right\|_{\Omega_i^{(m_i)}} &= \left\|f_{j*}^{(m_i)} - f_{jk*}^{(m_i)}\right\|_{\psi_{ij}(\Omega_i^{(m_i)})}\\
&\le \left\|f_{j*}^{(m_i)} - f_{j\,j+1*}^{(m_i)}\right\|_{\psi_{ij}(\Omega_i^{(m_i)})} + \dots + \left\|f_{k-1*}^{(m_i)} - f_{k-1\,k*}^{(m_i)}\right\|_{\psi_{i\,k-1*}^{(m_i)}(\Omega_i^{(m_i)})}\\
&\le \left\|f_{j*}^{(m_j)} - f_{j\,j+1*}^{(m_j)}\right\|_{\Omega_j^{(m_j)}} + \dots + \left\|f_{k-1*}^{(m_{k-1})} - f_{k-1\,k*}^{(m_{k-1})}\right\|_{\Omega_{k-1}^{(m_{k-1})}} \le \varepsilon_j + \dots \varepsilon_{k-1} < \bar\varepsilon_j
\end{aligned} \tag{6.2}$$

because

$$\psi_{ij*}^{(m_i)}(\Omega_i^{(m_i)}) \subset \psi_{ij*}^{(m_i)}(B_i''^{(m_i)}) \subset B_j'^{(m_j)} \subset \Omega_j^{(m_j)}$$

and $f_{jk*}^{(m_j)} = f_{jk*}^{(m_i)}$ on $\Omega_j^{(m_j)} \cap B_j^{(m_i)} \supset \psi_{ij*}^{(m_i)}(\Omega_i^{(m_i)})$.

Let $\mathcal{U}_i$ be a finite collection of charts of M_i with domains $U_{i,a}$, and let $\mathcal{K}_i = \{K_{i,a}\}$ be a family of compact subsets of M_i, with the same index set as $\mathcal{U}_i$, such that $K_{i,a} \subset U_{i,a}$ for all a, and $\overline{B}_i'' \subset \bigcup_a K_{i,a} =: K_i$. Thus $\Omega_i \subset K_i$. Choose some $C_i \ge 1$ satisfying (4.3) for $\mathbb{E}$-valued functions with $\mathcal{U}_i$, $\mathcal{K}_i$, Ω_i and g_i. With the notation of Section 4.2, for any $\rho > 0$ and $n+1 \le \mu \le 2^{m_i}n$, let $\sigma_{i,a,\rho,\mu}^{(m_i)} : U_{i,a} \to U_{i,a}^{(m_i)}$ be the section of each $\pi : U_{i,a}^{(m_i)} \to U_{i,a}$ of the type used in Lemma 4.2.2-(ii). Since $\Omega_i \subset \operatorname{Int}(\Omega_i^{(m_i)})$, there is some $\rho > 0$ so that $\sigma_{i,a,\rho,\mu}^{(m_i)}(K_{i,a} \cap \Omega_i) \subset K_{i,a}^{(m_i)} \cap \Omega_i^{(m_i)}$ for all a and μ. Thus, by Lemma 4.2.2-(ii), given any $\varepsilon > 0$, there is some $\delta > 0$, depending on ε and ρ, such that

$$\|f_{ij*}^{(m_i)} - f_{ik*}^{(m_i)}\|_{\Omega_i^{(m_i)}} < \delta \Longrightarrow \|f_{ij} - f_{ik}\|_{C^m,\Omega_i,\mathcal{U}_i,\mathcal{K}_i} < \varepsilon/C_i\,. \tag{6.3}$$

For j large enough, we have $\bar\varepsilon_j < \delta$, giving

$$\|f_{ij*}^{(m_i)} - f_{ik*}^{(m_i)}\|_{\Omega_i^{(m_i)}} < \delta \Longrightarrow \|f_{ij} - f_{ik}\|_{C^{m_i},\Omega_i,\mathcal{U}_i,\mathcal{K}_i} < \varepsilon/C_i \Longrightarrow \|f_{ij} - f_{ik}\|_{C^{m_i},\Omega_i,g_i} < \varepsilon$$

by (6.2), (6.3) and (4.3). This shows that $f_{ij}|_{\Omega_i}$ is a Cauchy sequence in the Banach space $C^{m_i}(\Omega_i,\mathbb{E})$ with $\|\ \|_{C^{m_i},\Omega_i,g_i}$, and therefore it is convergent. This completes the proof of Claim 6.2.1.

According to Claim 6.2.1, for each i, let $\hat f_{i\infty}=\lim_{k\to\infty}\hat f_k|_{\widehat\Omega_i}$ in $C^{m_i}(\widehat\Omega_i,\mathbb{E})$. Obviously, $\hat f_{j\infty}|_{\widehat\Omega_i}=\hat f_{i\infty}$ for $j>i$. Hence there is a function $\hat f\in C^\infty(\widehat M,\mathbb{E})$ whose restriction to every $\widehat\Omega_i$ is $\hat f_{i\infty}$. From (6.2), we get $\|\hat f^{(m_i)}_{i*}-\hat f^{(m_i)}_{k*}\|_{\widehat\Omega_i^{(m_i)}}<\bar\varepsilon_i$ for $k\ge i$, yielding $\|\hat f^{(m_i)}_{i*}-\hat f^{(m_i)}_{*}\|_{\widehat\Omega_i^{(m_i)}}\le\bar\varepsilon_i$. Hence $\psi_i:(M_i,f_i,x_i)\rightarrowtail(\widehat M,\hat x,\hat f)$ is an $(m_i,R'_i,\bar\lambda_i,\bar\varepsilon_i)$-pointed local quasi-equivalence. It follows that $([M_i,f_i,x_i],[\widehat M,\hat x,\hat f])\in\widehat U^{m_i}_{R'_i,s_i}$ for any sequence $s_i\downarrow 0$ so that $s_i>\max\{\ln\bar\lambda_i,\bar\varepsilon_i\}$, obtaining that $[M_i,f_i,x_i]\to[\widehat M,\hat x,\hat f]$ as $i\to\infty$ in $\widehat{\mathcal{M}}^\infty_*(n)$. □

Corollary 6.2.10. *$\widehat{\mathcal{M}}^\infty_*(n)$ is Polish.*

Proof. This is the content of Propositions 6.2.8 and 6.2.9 together. □

Corollaries 6.2.7 and 6.2.10 give Theorem 1.4.2.

6.3 Foliated structure of $\widehat{\mathcal{M}}^\infty_{*,\mathrm{imm}}(n)$

The properties stated in Theorem 1.4.3 are given by propositions of this section.

Proposition 6.3.1. *$\widehat{\mathcal{M}}^\infty_{*,\mathrm{imm}}(n)$ is Polish.*

Proof. For each $R>0$, let $\mathcal{W}_R\subset\widehat{\mathcal{M}}^\infty_*(n)$ be the open subset consisting of the points $[M,f,x]$ such that $f|_\Omega$ is an immersion for some compact domain $\Omega\subset M$ containing $B_M(x,R)$. Then $\widehat{\mathcal{M}}_{*,\mathrm{imm}}(n)=\bigcap_{R=1}^\infty\mathcal{W}_R$ is a G_δ in $\widehat{\mathcal{M}}^\infty_*(n)$. So $\widehat{\mathcal{M}}^\infty_{*,\mathrm{imm}}(n)$ is a Polish space by Corollary 6.2.10 and [51, Theorem I.3.11]. □

Proposition 6.3.2. *$\widehat{\mathcal{M}}^\infty_{*,\mathrm{imm,c}}(n)$ is dense in $\widehat{\mathcal{M}}^\infty_{*,\mathrm{c}}(n)$.*

Proof. With the notation of the proof of Proposition 6.2.8, $\widehat{\mathcal{M}}^\infty_{*,\mathrm{c}}(n)$ has an open partition consisting of the subspaces

$$\widehat{\mathcal{M}}^\infty_*(M)=\{\,[M,f,x]\mid f\in C^\infty(M,\mathbb{E}),\ x\in M\,\}\quad(M\in\mathcal{C})\ .$$

Thus it is enough to prove that each intersection $\widehat{\mathcal{M}}^\infty_*(M)\cap\widehat{\mathcal{M}}^\infty_{*,\mathrm{imm}}(n)$ is dense in $\widehat{\mathcal{M}}^\infty_*(M)$. This means that $C^\infty_{\mathrm{imm}}(M,\mathbb{E})$ is dense in $C^\infty(M,\mathbb{E})$, which follows easily from [44, Theorem 2.2.12]. □

Proposition 6.3.3. *There is a connected complete open Riemannian manifold N and some $h\in C^\infty_{\mathrm{imm}}(N,\mathbb{E})$ such that $\hat\imath_{N,h}$ is dense in $\widehat{\mathcal{M}}^\infty_{*,\mathrm{imm,o}}(n)$.*

Proof. In the proof of Proposition 6.2.8, we can assume that $\mathcal{F}_M \subset C^\infty_{\mathrm{imm}}(M,\mathbb{E})$ for each $M \in \mathcal{C}$ by [44, Theorem 2.2.12]. Then the set (6.1), denoted here by $\{\,[(M_i,g_i),f_i,x_i] \mid i \in \mathbb{N}\,\}$, is contained in $\widehat{\mathcal{M}}^\infty_{*,\mathrm{imm}}(n)$.

For every i, let $r_i = \max_{x\in M_i} d(x_i,x)$, and let $B_i = B_i(x_i,r_i/2)$ and $B'_i = B_i(x_i,2r_i/3)$. Let N be a C^∞ connected manifold obtained by modifying $\bigsqcup_i M_i$ on the complement of $\bigsqcup_i \overline{B'_i}$; for instance, we can take N equal to the C^∞ connected sum $M_0 \# M_1 \# \cdots$, constructed by removing balls in the sets $M_i \smallsetminus \overline{B'_i}$. Equip N with a complete Riemannian metric g_N whose restriction to each B_i is g_i. For instance, we can take $g_N = \lambda g' + \mu g''$, where $\{\lambda,\mu\}$ is a C^∞ partition of unity of N subordinated to the open covering $\{\bigsqcup_i B'_i, N \smallsetminus \bigsqcup_i \overline{B_i}\}$, g' is the combination of the metrics g_i on $\bigsqcup_i B'_i$, and g'' is any complete metric on N. Form [44, Theorems 2.1.1 and 2.2.12], it easily follows that there is some $h \in C^\infty_{\mathrm{imm}}(N,\mathbb{E})$ whose restriction to each B_i is f_i. It is easy to see that N and h satisfies the conditions of the statement. □

Remark 6.3.1. The versions of Propositions 6.3.2 and 6.3.3 with embeddings instead of immersions also hold by [44, Theorems 2.1.4 and 2.2.13].

To define foliated charts in $\widehat{\mathcal{M}}^\infty_{*,\mathrm{imm}}(n)$, fix some $e \in \mathbb{E}$, and some linear subspace, $V \subset \mathbb{E}$, of dimension n. Let $\Pi_V : \mathbb{E} \to V$ denote the orthogonal projection. For each complete connected Riemannian manifold M and any $f \in C^\infty_{\mathrm{imm}}(M,\mathbb{E})$, let $\chi_{M,f} = \chi_{V,e,M,f} : M \to V$ be the C^∞ map defined by $\chi_{M,f}(x) = \Pi_V(f(x) - e)$. Let $\chi = \chi_{V,e} : \mathcal{M}^\infty_{*,\mathrm{imm}}(n) \to V$ be defined by $\chi([M,f,x]) = \chi_{M,f}(x)$.

Lemma 6.3.4. *χ is continuous*

Proof. The map χ equals the following composite of continuous maps:

$$\mathcal{M}^\infty_{*,\mathrm{imm}}(n) \xrightarrow{\mathrm{ev}} \mathbb{E} \xrightarrow{-e} \mathbb{E} \xrightarrow{\Pi_V} V\,, \tag{6.4}$$

where the translation by $-e$ in $\mathbb{E}$ is also denoted by $-e$. □

Given $\rho,\sigma > 0$ and $\kappa > 1$, let $B = B_V(0,\sigma)$, and consider the following subsets of $\widehat{\mathcal{M}}^\infty_{*,\mathrm{imm}}(n)$:

- $\mathcal{N}_0 = \mathcal{N}_0(V,e,\rho,\kappa,\sigma)$ consists of the classes $[M,f,x] \in \widehat{\mathcal{M}}^\infty_{*,\mathrm{imm}}(n)$ such that $\chi_{M,f} : B_M(x,\tilde\rho) \to V$ is a $\tilde\kappa$-quasi-isometric embedding for some $\tilde\rho > 5\rho + \kappa\sigma$ and $\tilde\kappa \in (1,\kappa)$, and $\overline{B} \subset \chi_{M,f}(B_M(x,\rho))$.
- $\mathcal{N}_1 = \mathcal{N}_1(V,e,\rho,\kappa,\sigma)$ consists of the classes $[M,f,x] \in \widehat{\mathcal{M}}^\infty_{*,\mathrm{imm}}(n)$ that satisfy $[M,f,x'] \in \mathcal{N}_0$ for some $x' \in B_M(x,\rho)$.
- $\mathcal{N}_2 = \mathcal{N}_2(V,e,\rho,\kappa,\sigma) := \mathcal{N}_1 \cap \chi^{-1}(B)$.

Using [44, Lemma 2.1.3], it easily follows that, for each $i \in \{0,1,2\}$, the sets $\mathcal{N}_i(V,e,\rho,\kappa,\sigma)$ form an open covering of $\mathcal{M}^{\infty}_{*,\mathrm{imm}}(n)$ by varying (V,e,ρ,κ,σ).

Lemma 6.3.5. *$\chi_{M,f} : B_M(x,4\rho+\kappa\sigma) \to V$ is an embedding for all $[M,f,x] \in \mathcal{N}_1$.*

Proof. For each $[M,f,x] \in \mathcal{N}_1$, take some $x' \in B_M(x,\rho)$ so that $[M,f,x'] \in \mathcal{N}_0$. Then $B_M(x,4\rho+\kappa\sigma) \subset B_M(x',5\rho+\kappa\sigma)$ and $\chi_{M,f} : B_M(x',5\rho+\kappa\sigma) \to V$ is an embedding. □

Let $\mathcal{Z} = \mathcal{N}_1 \cap \chi^{-1}(0)$, which is closed in $\mathcal{N}_2$. For each $[M,f,x] \in \mathcal{N}_1$, there is some $x' \in B_M(x,\rho)$ so that $[M,f,x'] \in \mathcal{N}_0$. Then there is some $x'' \in B_M(x',\rho)$ such that $\chi_{M,f}(x'') = 0$. Observe that $[M,f,x''] \in \mathcal{N}_1$, and therefore $[M,f,x''] \in \mathcal{Z}$. By Lemma 6.3.5, x'' is the unique point in $B_M(x,2\rho)$ such that $\chi_{M,f}(x'') = 0$. Thus the class $[M,f,x'']$ depends only on $[M,f,x]$. So a map $\Theta : \mathcal{N}_1 \to \mathcal{Z}$ is well defined by setting $\Theta([M,f,x]) = [M,f,x'']$.

Lemma 6.3.6. *Θ is continuous.*

Proof. Consider a convergent sequence $[M_i,f_i,x_i] \to [M,f,x]$ in $\mathcal{N}_1$. Take points $x'_i \in B_i(x_i,2\rho)$ and $x' \in B_M(x',2\rho)$ such that $\chi_{M_i,f_i}(x'_i) = \chi_{M,f}(x') = 0$. Thus $\Theta([M_i,f_i,x_i]) = [M_i,f_i,x'_i]$ and $\Theta([M,f,x]) = [M,f,x']$.

Given $m \in \mathbb{N}$ and $R,r > 0$, for i large enough, there is an $(m,R,\lambda_i,\varepsilon_i)$-pointed local quasi-equivalence $\phi_i : (M,f,x) \rightarrowtail (M_i,f_i,x_i)$ for some $\lambda_i \in (1,e^r)$ and $\varepsilon_i \in (0,r)$. Suppose that $R > 3\rho$ and $e^r < 3/2$; in particular, $\overline{B}_M(x,3\rho) \subset \operatorname{dom}\phi_i$.

Claim 6.3.1. $B_i(x_i,2\rho) \subset \phi_i(B_M(x,3\rho))$.

The set $A = B_i(x_i,2\rho) \cap \phi_i(B_M(x,3\rho))$ contains x_i and is open in the connected space $B_i(x_i,2\rho)$. Then Claim 6.3.1 follows by showing that A is also closed in $B_i(x_i,2\rho)$. This holds since $A = B_i(x_i,2\rho) \cap \phi_i(\overline{B}_M(x,3\rho))$ because, for every $y \in M$ with $d_M(x,y) = 3\rho$, we have

$$d_i(x_i,\phi_i(y)) \geq \frac{1}{\lambda_i} d_M(x,y) > 3\rho e^{-r} > 2\rho\ .$$

According to Claim 6.3.1, there is some $\bar{x}'_i \in B_M(x,3\rho)$ such that $\phi_i(\bar{x}'_i) = x'_i$. We have

$$\begin{aligned} d_M(x',\bar{x}'_i) &\leq \kappa\|\chi_{M,f}(x') - \chi_{M,f}(\bar{x}'_i)\| = \kappa\|\chi_{M,f}(\bar{x}'_i) - \chi_{M_i,f_i}(x'_i)\| \\ &\leq \kappa\|f(\bar{x}'_i) - f_i(x'_i)\| = \kappa\|f(\bar{x}'_i) - f_i\circ\phi(\bar{x}'_i)\| < \kappa\varepsilon_i < \kappa r\ . \end{aligned}$$

Therefore, by the continuity of $\hat{\imath}_{M,f}$, for any $S,s > 0$, if r is small enough and i large enough, there is an (m,S,μ_i,δ_i)-pointed local quasi-equivalence $\psi_i : (M,f,x') \rightarrowtail (M,f,\bar{x}'_i)$ with $\mu_i \in (1,e^{s/2})$ and $\delta_i \in (0,s/2)$. On the other hand, observe that $\phi_i : (M,\bar{x}'_i,f) \rightarrowtail (M_i,f_i,x'_i)$ is an $(m,R-2\rho,\lambda_i,\varepsilon_i)$-pointed local quasi-equivalence. Hence, if moreover $R > e^{s/2}S + 2\rho$ and $r < s/2$, we get that $\phi_i \circ \psi_i : (M, f, x') \rightarrowtail (M_i, f_i, x'_i)$ is an $(m, S, \mu_i\lambda_i, \delta_i + \varepsilon_i)$-pointed local quasi-equivalence with $\mu_i\lambda_i \in (1, e^s)$ and $\delta_i + \varepsilon_i \in (0, s)$ by Lemma 6.1.5-(i). This shows that $[M_i, f_i, x'_i] \to [M, f, x']$ in $\widehat{\mathcal{M}}^{\infty}_{*}(n)$. $\square$

Let $\Phi = (\chi, \Theta) : \mathcal{N}_2 \to B \times \mathcal{Z}$.

Lemma 6.3.7. *Φ is bijective, and $\Phi^{-1}(v, [M, f, x]) = [M, f, x']$ for each $(v, [M, f, x]) \in B \times \mathcal{Z}$, where x' is the unique point in $B_M(x, 2\rho) \cap \chi^{-1}_{M,f}(v)$.*

Proof. To prove that Φ is injective, let $[M_i, f_i, x_i] \in \mathcal{N}_2$ $(i \in \{1, 2\})$ be such that $\Phi([M_1, f_1, x_1]) = \Phi([M_2, f_2, x_2])$; i.e., $\chi_{M_1,f_1}(x_1) = \chi_{M_2,f_2}(x_2)$ and $[M_1, f_1, x'_1] = [M_2, f_2, x'_2]$ for points $x'_i \in B_i(x_i, 2\rho)$ with $\chi_{M_i,f_i}(x'_i) = 0$. Thus there is a pointed equivalence $\phi : (M_1, f_1, x'_1) \to (M_2, f_2, x'_2)$. We get $\phi(x_1) = x_2$ because $\chi_{M_2,f_2} \circ \phi(x_1) = \chi_{M_1,f_1}(x_1) = \chi_{M_2,f_2}(x_2)$, the map $\chi_{M_i,f_i} : (B_i(x'_i, 2\rho), x'_i) \to (V, 0)$ is a pointed embedding (Lemma 6.3.5), and $x_i \in B_i(x'_i, 2\rho)$. So $\phi : (M_1, f_1, x_1) \to (M_2, f_2, x_2)$ is a pointed equivalence, and therefore $[M_1, f_1, x_1] = [M_2, f_2, x_2]$.

Now, let us prove that Φ is surjective, showing the stated expression of Φ^{-1}. Let $(v, [M, f, x]) \in B \times \mathcal{Z}$. There is some $y \in B_M(x, \rho)$ such that $[M, f, y] \in \mathcal{N}_0$. So there is some $x' \in B_M(y, \rho)$ such that $\chi_{M,f}(x') = v$. It follows that $[M, f, x'] \in \mathcal{N}_1$, $\Theta([M, f, x']) = [M, f, x]$ and $\chi([M, f, x']) = v$. Therefore $[M, f, x'] \in \mathcal{N}_2$ and $\Phi([M, f, x']) = (v, [M, f, x])$. Moreover x' is the unique point in $B_M(x, 2\rho) \cap \chi^{-1}_{M,f}(v)$ by Lemma 6.3.5. $\square$

Lemma 6.3.8. *Φ^{-1} is continuous.*

Proof. Consider a convergent sequence $(v_i, [M_i, f_i, x_i]) \to (v, [M, f, x])$ in $B \times \mathcal{Z}$. Take points $x'_i \in B_i(x_i, 2\rho)$ and $x' \in B_M(x, 2\rho)$ such that $\chi_{M_i,f_i}(x'_i) = v_i$ and $\chi_{M,f}(x') = v$. Thus $\Phi^{-1}(v_i, [M_i, f_i, x_i]) = [M_i, f_i, x'_i]$ and $\Phi^{-1}(v, [M, f, x]) = [M, f, x']$.

Given $m \in \mathbb{N}$ and $R, r > 0$, if i is large enough, then $\|v - v_i\| < r$, and there is an $(m, R, \lambda_i, \varepsilon_i)$-pointed local quasi-equivalence $\phi_i : (M, f, x) \rightarrowtail (M_i, f_i, x_i)$ for some $\lambda_i \in (1, e^r)$ and $\varepsilon_i \in (0, r)$. Suppose that $R > 3\rho$ and $e^r < 3/2$; in particular, $\overline{B}_M(x, 3\rho) \subset \operatorname{dom} \phi_i$. Like in Claim 6.3.1, we get $B_i(x_i, 2\rho) \subset \phi_i(B_M(x, 3\rho))$. Then, since $x'_i \in B_i(x_i, 2\rho)$, there is some $\bar{x}'_i \in B_M(x, 3\rho)$ such that $\phi_i(\bar{x}'_i) = x'_i$. We have

$$\begin{aligned} d_M(x', \bar{x}'_i) &\le \kappa \|\chi_{M,f}(x') - \chi_{M,f}(\bar{x}'_i)\| \le \kappa \left(\|\chi_{M,f}(\bar{x}'_i) - \chi_{M_i,f_i}(x'_i)\| + \|v - v_i\|\right) \\ &< \kappa \left(\|f(\bar{x}'_i) - f_i(x'_i)\| + r\right) = \kappa \left(\|f(\bar{x}'_i) - f_i \circ \phi(\bar{x}'_i)\| + r\right) < \kappa(\varepsilon_i + r) < 2\kappa r \,. \end{aligned}$$

Hence we get $[M_i, f_i, x'_i] \to [M, f, x']$ in $\widehat{\mathcal{M}}^{\infty}_{*}(n)$ like in the end of the proof of Lemma 6.3.6. $\square$

Corollary 6.3.9. *Φ is a homeomorphism.*

Proof. This follows from Lemmas 6.3.4, 6.3.6, 6.3.7 and 6.3.8. □

Lemma 6.3.10. *If $[M,f,x]\in\chi^{-1}(B)$ and $[M,f,x']\in\mathcal{Z}$ for some $x'\in B_M(x,2\rho)$, then $[M,f,x]\in\mathcal{N}_2$.*

Proof. Let $v=\chi([M,f,x])\in B$. By Lemma 6.3.7, there is some $x''\in B_M(x',2\rho)$ be such that $[M,f,x'']\in\mathcal{N}_2$ and $\Phi([M,f,x''])=(v,[M,f,x'])$. Then $x=x''$ by Lemma 6.3.5 applied to $\chi_{M,f}:B_M(x',2\rho)\to V$. □

Take $(\widetilde{V},\tilde{e},\tilde{\rho},\tilde{\kappa},\tilde{\sigma})$ like (V,e,ρ,κ,σ). Let $\widetilde{\mathcal{N}}_i=\mathcal{N}_i(\widetilde{V},\tilde{e},\tilde{\rho},\tilde{\kappa},\tilde{\sigma})$ for $i\in\{0,1,2\}$, and let $\widetilde{\Phi}=(\widetilde{\chi},\widetilde{\Theta}):\widetilde{\mathcal{N}}_2\to\widetilde{B}\times\widetilde{\mathcal{Z}}$ be defined like $\Phi=(\chi,\Theta):\mathcal{N}_2\to B\times\mathcal{Z}$, using $(\widetilde{V},\tilde{e},\tilde{\rho},\tilde{\kappa},\tilde{\sigma})$. Moreover, for each $[M,f,x]\in\widehat{\mathcal{M}}_*^\infty(n)$, let $\widetilde{\chi}_{M,f}:M\to\widetilde{V}$ be defined like $\chi_{M,f}:M\to V$, using $\Pi_{\widetilde{V}}$ and $\tilde{e}$. Suppose that $\mathcal{N}_2\cap\widetilde{\mathcal{N}}_2\neq\emptyset$, and consider the map $\widetilde{\Phi}\circ\Phi^{-1}:\Phi(\mathcal{N}_2\cap\widetilde{\mathcal{N}}_2)\to\widetilde{\Phi}(\mathcal{N}_2\cap\widetilde{\mathcal{N}}_2)$.

Lemma 6.3.11. *Let $(v,[M,f,x])\in\Phi(\mathcal{N}_2\cap\widetilde{\mathcal{N}}_2)$. Then $\widetilde{\Phi}\circ\Phi^{-1}(v,[M,f,x])=(\tilde{v},[M,f,\tilde{x}])$, where $\tilde{x}\in\widetilde{\chi}_{M,f}^{-1}(0)$ is determined by the condition*

$$B_M(x,2\rho)\cap B_M(\tilde{x},2\tilde{\rho})\cap\chi_{M,f}^{-1}(B)\cap\widetilde{\chi}_{M,f}^{-1}(\widetilde{B})\neq\emptyset\,, \tag{6.5}$$

and $\tilde{v}$ is the image of v by the composite

$$\chi_{M,f}(O)\xrightarrow{\chi_{M,f}^{-1}}O\xrightarrow{\widetilde{\chi}_{M,f}}\widetilde{\chi}_{M,f}(O)\,, \tag{6.6}$$

where $O=B_M(x,2\rho)\cap B_M(\tilde{x},2\tilde{\rho})$.

Proof. Let $[M,f,x']\in\mathcal{N}_2\cap\widetilde{\mathcal{N}}_2$ such that $\Phi([M,f,x'])=(v,[M,f,x])$ and $\widetilde{\Phi}([M,f,x'])=(\tilde{v},[M,f,\tilde{x}])$. By Lemma 6.3.7, this means that $\chi_{M,f}(x)=\widetilde{\chi}_{M,f}(\tilde{x})=0$, $x'\in B_M(x,2\rho)\cap B_M(\tilde{x},2\tilde{\rho})$, $\chi_{M,f}(x')=v$ and $\widetilde{\chi}_{M,f}(x')=\tilde{v}$, obtaining (6.5) and (6.6). Note that (6.6) makes sense by Lemma 6.3.5.

Now, assume that (6.5) also holds using another point $\tilde{y}\in\widetilde{\chi}_{M,f}^{-1}(0)$ instead of $\tilde{x}$. Thus there is some $y'\in B_M(x,2\rho)\cap B_M(\tilde{x},2\tilde{\rho})$ with $w:=\chi_{M,f}(y')\in B$ and $\tilde{w}:=\widetilde{\chi}_{M,f}(y')=\widetilde{B}$. Then $[M,f,y']\in\mathcal{N}_2$ by Lemma 6.3.10, and $\Phi([M,f,y'])=(w,[M,f,x])$ and $\widetilde{\Phi}([M,f,y'])=(w,[M,f,\tilde{y}])$. We have

$$d_M(\tilde{x},\tilde{y})\le d_M(\tilde{x},x')+d_M(x',y')+d_M(y',\tilde{y})<4\tilde{\rho}+\tilde{\kappa}\|\tilde{v}-\tilde{w}\|<4\tilde{\rho}+\tilde{\kappa}\tilde{\sigma}\,.$$

Since moreover $\tilde{\chi}_{M,f}(\tilde{x})=0=\tilde{\chi}_{M,f}(\tilde{y})$, we get $\tilde{x}=\tilde{y}$ by Lemma 6.3.5. □

Proposition 6.3.12. *All possible maps $\Phi:\mathcal{N}_2\to B\times\mathcal{Z}$ form an atlas of a C^∞ foliated structure on $\widehat{\mathcal{M}}_{*,\mathrm{imm}}^\infty(n)$.*

Proof. The maps $\Phi : \mathcal{N}_2 \to B \times \mathcal{Z}$ are homeomorphisms (Corollary 6.3.9). All possible sets $\mathcal{N}_2$ form an open cover of $\widehat{\mathcal{M}}^{\infty}_{*,\mathrm{imm}}(n)$. Moreover, in Lemma 6.3.11, it follows from (6.5) that $[M, f, \tilde{x}]$ depends only on $[M, f, x]$. Thus all possible maps $\Phi : \mathcal{N}_2 \to B \times \mathcal{Z}$ form an atlas of a foliated structure on $\widehat{\mathcal{M}}^{\infty}_{*,\mathrm{imm}}(n)$.

With the notation of Lemma 6.3.11 and the terminology of Section 4.1, it only remains to show that $\widetilde{\Phi} \circ \Phi^{-1}$ is C^∞; i.e., to prove that the mapping $(v, [M, f, x]) \mapsto \tilde{v}$ is C^∞. First, note that, for each fixed $[M, f, x]$, the mapping $v \mapsto \tilde{v}$ is C^∞ because (6.6) is C^∞. Consider now a convergent sequence $[M_i, f_i, x_i] \to [M, f, x]$ in $\mathcal{Z}$. Let $\tilde{x}_i \in \widetilde{\chi}^{-1}_{M_i,f_i}(0)$ be determined by

$$B_M(x_i, 2\rho) \cap B_M(\tilde{x}_i, 2\tilde{\rho}) \cap \chi^{-1}_{M_i,f_i}(B) \cap \widetilde{\chi}^{-1}_{M_i,f_i}(\widetilde{B}) \neq \emptyset ,$$

and let $O_i = B_M(x_i, 2\rho) \cap B_M(\tilde{x}, 2\tilde{\rho})$. Given $m \in \mathbb{N}$ and $R, r > 0$, for each i large enough, there is an $(m, R, \lambda_i, \varepsilon_i)$-pointed local quasi-equivalence $\phi_i : (M_i, f_i, x_i) \rightarrowtail (M, f, x)$ for some $\lambda_i \in (1, e^r)$ and $\varepsilon_i \in (0, r)$. Let $\Omega^{(m)}_i$ be a compact domain in $\operatorname{dom} \phi^{(m)}_{i*}$ such that $B^{(m)}_i(x_i, R) \subset \Omega^{(m)}_i$ and $\phi^{(m)}_{i*} : (\Omega^{(m)}_i, f^{(m)}_{i*}) \to (T^{(m)}M, f^{(m)}_*)$ is an $(\varepsilon_i, \lambda_i)$-quasi-equivalence. Since $(\Pi_V)^{(m)}_* \equiv \Pi_{V^{2^m}} : T^{(m)}\mathbb{E} \equiv \mathbb{E}^m \to T^{(m)}V \equiv V^{2^m}$, we have

$$\begin{aligned}\left\|\chi^{(m)}_{M_i,f_i*} - (\chi_{M,f} \circ \phi_i)^{(m)}_*\right\|_{\Omega^{(m)}_i} &\leq \left\|(f_i - e)^{(m)}_* - ((f - e) \circ \phi_i)^{(m)}_*\right\|_{\Omega^{(m)}_i} \\ &= \left\|f^{(m)}_{i*} - (\phi^*_i f)^{(m)}_*\right\|_{\Omega^{(m)}_i} < \varepsilon_i < r . \quad (6.7)\end{aligned}$$

Assume that $R > 2e^r\rho$. Then $B_i(x_i, 2\rho) \subset B_i(x_i, R)$, and, like in Claim 6.3.1, we also get $B_M(x, 2\rho) \subset \phi_i(B_i(x_i, R))$. Thus $O_i \subset B_i(x_i, R)$ and $O \subset \phi_i(B_i(x_i, R))$. Let $\Xi \subset \chi_{M,f}(O)$ be a compact domain, which is also contained in $\chi_{M_i,f_i}(O_i)$ for i large enough. Let $\Xi^{(m)}$ be a compact domain contained in $T^{(m)}\mathbb{E}$ such that $\Xi \subset \operatorname{Int}(\Xi^{(m)})$ and

$$(\chi^{-1}_{M_i,f_i})^{(m)}_*(\Xi^{(m)}) \subset \Omega^{(m)}_i \cap T^{(m)}O_i , \quad (\chi^{-1}_{M,f})^{(m)}_*(\Xi^{(m)}) \subset \phi^{(m)}_{i*}(\Omega^{(m)}_i) \cap T^{(m)}O .$$

Since the restrictions of $(\chi^{-1}_{M,f})^{(m)}_*$ and $(\widetilde{\chi}_{M,f})^{(m)}_*$ to the respective compact domains $\Xi^{(m)}$ and $\chi^{-1}_{M,f}(\Xi^{(m)}) \cap T^{(m)}O$ are C^∞ embeddings, these restrictions are ν-quasi-isometric for some $\nu \geq 1$. Hence, by (6.7),

$$\begin{aligned}d^{(m)}_i\left((\phi_i \circ \chi^{-1}_{M_i,f_i})^{(m)}_*(\xi), (\chi^{-1}_{M,f})^{(m)}_*(\xi)\right) &\leq \nu \left\|(\chi_{M,f} \circ \phi_i \circ \chi^{-1}_{M_i,f_i})^{(m)}_*(\xi) - \xi\right\| \\ &= \nu \left\|(\chi_{M,f} \circ \phi_i \circ \chi^{-1}_{M_i,f_i})^{(m)}_*(\xi) - (\chi_{M_i,f_i} \circ \chi^{-1}_{M_i,f_i})^{(m)}_*(\xi)\right\| < \nu r , \quad (6.8)\end{aligned}$$

for all $\xi \in \Xi^{(m)}$. On the other hand, like in (6.7), we get

$$\left\|\widetilde{\chi}^{(m)}_{M_i,f_i*} - (\widetilde{\chi}_{M,f} \circ \phi_i)^{(m)}_*\right\|_{\Omega^{(m)}_i} < r . \quad (6.9)$$

Combining (6.8) and (6.9), we obtain the following for all $\xi \in \Xi^{(m)}$:

$$
\begin{aligned}
\big\|(\widetilde{\chi}_{M_i,f_i} \circ \chi_{M_i,f_i}^{-1})_*^{(m)}(\xi) - (\widetilde{\chi}_{M,f} \circ \chi_{M,f}^{-1})_*^{(m)}(\xi)\big\| \\
\le \big\|(\widetilde{\chi}_{M_i,f_i} \circ \chi_{M_i,f_i}^{-1})_*^{(m)}(\xi) - (\widetilde{\chi}_{M,f} \circ \phi_i \circ \chi_{M_i,f_i}^{-1})_*^{(m)}(\xi)\big\| \\
+ \big\|(\widetilde{\chi}_{M,f} \circ \phi_i \circ \chi_{M_i,f_i}^{-1})_*^{(m)}(\xi) - (\widetilde{\chi}_{M,f} \circ \chi_{M,f}^{-1})_*^{(m)}(\xi)\big\| \\
< r + \nu\, d_i^{(m)}\left((\phi_i \circ \chi_{M_i,f_i}^{-1})_*^{(m)}(\xi), (\chi_{M,f}^{-1})_*^{(m)}(\xi)\right) < (1+\nu^2)r\ .
\end{aligned}
$$

Note that the same choices of Ξ and $\Xi^{(m)}$ are valid for all r small enough, obtaining that $(\widetilde{\chi}_{M_i,f_i} \circ \chi_{M_i,f_i}^{-1})_*^{(m)} \to (\widetilde{\chi}_{M,f} \circ \chi_{M,f}^{-1})_*^{(m)}$ uniformly on $\Xi^{(m)}$. Moreover the same choice of Ξ is valid for all m, and therefore $\widetilde{\chi}_{M_i,f_i} \circ \chi_{M_i,f_i}^{-1} \to \widetilde{\chi}_{M,f} \circ \chi_{M,f}^{-1}$ on Ξ with respect to the C^∞ topology by the obvious version of Lemma 4.2.2 for maps between open subsets of $\mathbb{R}^n$. Since every point in $\chi_{M,f}(O)$ belongs to some domain Ξ as above if r is chosen small enough, it follows that $\widetilde{\Phi} \circ \Phi^{-1}$ is C^∞. $\square$

Now, let $\widehat{\mathfrak{F}}_{*,\mathrm{imm}}^\infty(n)$ denote the C^∞ foliated structure on $\widehat{\mathcal{M}}_{*,\mathrm{imm}}^\infty(n)$ defined by the maps Φ according to Proposition 6.3.12.

Proposition 6.3.13. *The following properties hold:*

(i) $\widehat{\mathfrak{F}}_{*,\mathrm{imm}}^\infty(n)$ *is the unique* C^∞ *foliated structure on* $\widehat{\mathcal{M}}_{*,\mathrm{imm}}^\infty(n)$ *such that its underlying topological foliated structure is* $\widehat{\mathfrak{F}}_{*,\mathrm{imm}}(n)$ *and* $\mathrm{ev} : \widehat{\mathcal{M}}_{*,\mathrm{imm}}^\infty(n) \to \mathbb{E}$ *is a* C^∞ *immersion.*

(ii) *For each* $[M,f,x] \in \widehat{\mathcal{M}}_{*,\mathrm{imm}}^\infty(n)$, *the map* $\hat\imath_{M,f} : M \to \operatorname{im}\hat\imath_{M,f}$ *is a local diffeomorphism, where the leaf* $\operatorname{im}\hat\imath_{M,f}$ *is equipped with the* C^∞ *structure induced by* $\widehat{\mathfrak{F}}_{*,\mathrm{imm}}^\infty(n)$.

Proof. Take a foliated chart $\Phi : \mathcal{N}_2 \to B \times \mathcal{Z}$ as above. For each $[M,f,x] \in \mathcal{Z}$, the restriction of $\mathrm{ev} \circ \Phi^{-1}$ to $B \times \{[M,f,x]\} \equiv B$ is the composite

$$
B \xrightarrow{\chi_{M,f}^{-1}} B_M(x,2\rho) \cap \chi_{M,f}^{-1}(B) \xrightarrow{f} \mathbb{E}\ ,
$$

where the first map is a C^∞ diffeomorphism, and the second one is a C^∞ immersion. Take a convergent sequence $[M_i,f_i,x_i] \to [M,f,x]$ in $\mathcal{Z}$, and let $\Xi \subset B$ be any compact domain. Given $R > 2\rho$ and a compact domain $\Omega \subset M$ containing $B_M(x,R)$, there is a C^∞ pointed embedding $\phi_i : (\Omega,x) \to (M_i,x_i)$ for i large enough such that $\phi_i^* g_i \to g_M$ and $\phi_i^* f_i \to f$ on Ω with respect to the C^∞ topology. So $B_i(x_i,R) \subset \phi_i(\Omega)$ for i large enough. Thus also $\phi_i^*\chi_{M_i,f_i} \to \chi_{M,f}$ on Ω with respect to the C^∞ topology, and therefore $\phi_i^{-1} \circ \chi_{M_i,f_i}^{-1} \to \chi_{M,f}^{-1}$ on Ξ with respect to the C^∞ topology [44, p. 64, Exercise 9]. Hence

$$
f_i \circ \chi_{M_i,f_i}^{-1} - f \circ \chi_{M,f}^{-1} = f_i \circ \phi_i \circ (\phi_i^{-1} \circ \chi_{M_i,f_i}^{-1} - \chi_{M,f}^{-1}) + (f_i \circ \phi_i - f) \circ \chi_{M,f}^{-1} \to 0
$$

on Ξ with respect to the C^∞ topology. Since any element of B is contained some Ξ as above, it follows that $\mathrm{ev}\circ\Phi^{-1}$ is a C^∞ immersion, and therefore $\mathrm{ev}:\widehat{\mathcal{M}}^{\infty}_{*,\mathrm{imm}}(n)\to\mathbb{E}$ is C^∞ with respect to $\widehat{\mathcal{M}}^{\infty}_{*,\mathrm{imm}}(n)$. This shows (i), except uniqueness.

According to Lemma 6.3.7, for each chart $\Phi:\mathcal{N}_2\to B\times\mathcal{Z}$, the plaque that corresponds to each $[M,f,x]\in\mathcal{Z}$ is $\hat\iota_{M,f}(B_M(x,2\rho)\cap\chi_{M,f}^{-1}(B))$. Moreover the composite

$$B_M(x,2\rho)\cap\chi_{M,f}^{-1}(B)\xrightarrow{\hat\iota_{M,f}}\hat\iota_{M,f}(B_M(x,2\rho)\cap\chi_{M,f}^{-1}(B))\xrightarrow{\chi}B$$

is the diffeomorphism $\chi_{M,f}:B_M(x,2\rho)\cap\chi_{M,f}^{-1}(B)\to B$. This shows that the leaf topology on $\widehat{\mathcal{M}}_{*,\mathrm{imm}}(n)$ equals the topological sum of all possible spaces $\mathrm{im}\,\hat\iota_{M,f}$ with the topology so that $\hat\iota_{M,f}:M\to\mathrm{im}\,\hat\iota_{M,f}$ is a local homeomorphism, obtaining that these spaces are the leaves because they are connected. It also follows that $\hat\iota_{M,f}:M\to\mathrm{im}\,\hat\iota_{M,f}$ is a local diffeomorphism for each leaf $\mathrm{im}\,\hat\iota_{M,f}$. This shows (ii).

Now, suppose $\mathrm{ev}:\widehat{\mathcal{M}}^{\infty}_{*,\mathrm{imm}}(n)\to\mathbb{E}$ is C^∞ with respect to some C^∞ foliated structure $\mathcal{G}$ whose underlying topological foliated structure is $\widehat{\mathcal{F}}_{*,\mathrm{imm}}(n)$. Then $\chi:\widehat{\mathcal{M}}^{\infty}_{*,\mathrm{imm}}(n)\to V$ is also C^∞ with respect to $\mathcal{G}$ because it equals the composite (6.4). So each chart $\Phi=(\chi,\Theta):\mathcal{N}_2\to B\times\mathcal{Z}$ is also C^∞ with respect to $\mathcal{G}$ and the C^∞ product foliated structure of $B\times\mathcal{Z}$. Moreover, for all complete connected Riemannian manifold M and $f\in C^\infty_{\mathrm{imm}}(M)$, the map $\hat\iota_{M,f}:M\to\mathrm{im}\,\hat\iota_{M,f}$ is a C^∞ local diffeomorphism with respect to the C^∞ structure induced by $\mathcal{G}$ on the leaf $\mathrm{im}\,\hat\iota_{M,f}$ because ev is a C^∞ immersion and $\mathrm{ev}\circ\hat\iota_{M,f}=f$, which is a C^∞ local embedding. Thus the restriction of $\chi:\mathcal{N}_2\to B$ to each plaque is a C^∞ diffeomorphism. Using again [44, p. 64, Exercise 9], it follows that $\Phi:\mathcal{N}_2\to B\times\mathcal{Z}$ is also C^∞ foliated diffeomorphism with respect to the restriction of $\mathcal{G}$ and the C^∞ product foliated structure of $B\times\mathcal{Z}$. This shows that $\mathcal{G}=\widehat{\mathcal{F}}^{\infty}_{*,\mathrm{imm}}(n)$, completing the proof of (i). □

Consider a leaf $\mathrm{im}\,\hat\iota_{M,f}$ of $\widehat{\mathcal{F}}^{\infty}_{*,\mathrm{imm}}(n)$. Every $x\in M$ has an open neighborhood U in M so that $f:U\to\mathbb{E}$ is an embedding, obtaining that $\phi(U)\cap U=\emptyset$ for all $\phi\in\mathrm{Iso}(M,f)\smallsetminus\{\mathrm{id}_M\}$. Therefore the subgroup $\mathrm{Iso}(M,f)\subset\mathrm{Iso}(M)$ is discrete, the quotient projection $M\to\mathrm{Iso}(M,f)\backslash M$ is a covering map, and there is a unique Riemannian structure on the manifold $\mathrm{Iso}(M,f)\backslash M$ so that $M\to\mathrm{Iso}(M,f)\backslash M$ is a local isometry. Moreover $\hat\iota_{M,f}:M\to\mathrm{im}\,\hat\iota_{M,f}$ induces a diffeomorphism $\bar\iota_{M,f}:\mathrm{Iso}(M,f)\backslash M\to\mathrm{im}\,\hat\iota_{M,f}$. Thus $\hat\iota_{M,f}:M\to\mathrm{im}\,\hat\iota_{M,f}$ is a covering map, and $\mathrm{im}\,\hat\iota_{M,f}$ has a unique Riemannian metric so that $\hat\iota_{M,f}:M\to\mathrm{im}\,\hat\iota_{M,f}$ is a local isometry, and therefore $\bar\iota_{M,f}:\mathrm{Iso}(M,f)\backslash M\to\mathrm{im}\,\hat\iota_{M,f}$ becomes an isometry.

Proposition 6.3.14. *The above Riemannian metrics on the leaves of $\widehat{\mathcal{F}}^{\infty}_{*,\mathrm{imm}}(n)$ form a C^∞ Riemannian metric on $(\widehat{\mathcal{M}}^{\infty}_{*,\mathrm{imm}}(n),\widehat{\mathcal{F}}^{\infty}_{*,\mathrm{imm}}(n))$.*

Proof. Let $\Phi = (\chi, \Theta) : \mathcal{N}_2 \to B \times \mathcal{Z}$ be defined by any choice of $(V, e, \rho, \kappa, \sigma)$ as above, and let $[M_i, f_i, x_i] \to [M, f, x]$ be a convergent sequence in $\mathcal{Z}$. Let $\bar{g}_M$ and $\bar{g}_i$ be the metrics on B that correspond to g_M and g_i by the diffeomorphisms

$$\chi_{M,f} : P := B_M(x, 2\rho) \cap \chi_{M,f}^{-1}(B) \to B \;, \quad \chi_{M_i,f_i} : P_i := B_i(x_i, 2\rho) \cap \chi_{M_i,f_i}^{-1}(B) \to B \;,$$

respectively (see Lemma 6.3.7). According to the proof of Proposition 6.3.13-(ii), we have to prove that $\bar{g}_i \to \bar{g}_M$ as $i \to \infty$ in the weak C^∞ topology.

Given $m \in \mathbb{N}$, $R, r > 0$, for each i large enough, there is an $(m, R, \lambda_i, \varepsilon_i)$-pointed local quasi-equivalence $\phi_i : (M, f, x) \rightarrowtail (M_i, f_i, x_i)$ for some $\lambda_i \in (1, e^r)$ and $\varepsilon_i \in (0, r)$. Assuming $R > 2e^r\rho$, we get $B_M(x, 2\rho) \subset B_M(x, R)$ and $B_i(x_i, 2\rho) \subset \phi_i(B_M(x, R))$, like in the proof of Proposition 6.3.12. Take a compact domain $\Omega_i^{(m)} \subset \operatorname{dom} \phi_{i*}^{(m)}$ such that $B_i^{(m)}(x_i, R) \subset \Omega_i^{(m)}$ and $\phi_{i*}^{(m)} : \Omega_i^{(m)} \to T^{(m)}M$ is a $(\lambda_i, \varepsilon_i)$-quasi-isometry. Let $\Xi \subset B$ be a compact domain, and let $\Xi^{(m)}$ be a compact domain contained in $T^{(m)}B$ such that $\Xi \subset \operatorname{Int}(\Xi^{(m)})$ and

$$(\chi_{M,f}^{-1})_*^{(m)}(\Xi^{(m)}) \cap T^{(m)}P \subset \Omega_i^{(m)} \;, \quad (\chi_{M_i,f_i}^{-1})_*^{(m)}(\Xi^{(m)}) \cap T^{(m)}P_i \subset \phi_{i*}^{(m)}(\Omega_i^{(m)}) \;.$$

Like in (6.8), there is some $\nu \geq 1$, independent of i, such that

$$d_i^{(m)}\left((\phi_i^{-1} \circ \chi_{M_i,f_i}^{-1})_*^{(m)}(\xi), (\chi_{M,f}^{-1})_*^{(m)}(\xi)\right) < \nu r \;,$$

for all $\xi \in \Xi^{(m)}$. Since the choice of $\Xi^{(m)}$ is valid for all r small enough, it follows that $\phi_i^{-1} \circ \chi_{M_i,f_i}^{-1} \to \chi_{M,f}^{-1}$ in $C^m(\Xi, M)$ by the obvious version of Lemma 4.2.2 for maps between manifolds. Since the choice of Ξ is valid for all m, it follows that this convergence also holds in $C^\infty(\Xi, M)$. Take a compact domain $\Omega \subset M$ such that $B_M(x, R) \subset \Omega$ and $\phi_i^* g_i \to g_M$ on Ω with respect to the C^∞ topology. We get

$$(\phi_i^{-1} \circ \chi_{M_i,f_i}^{-1})^*(\phi_i^* g_i - g_M) \to (\chi_{M,f}^{-1})^* 0 = 0$$

on Ξ with respect to the C^∞ topology. So

$$\begin{aligned} \bar{g}_i - \bar{g}_M &= (\chi_{M_i,f_i}^{-1})^* g_i - (\chi_{M,f}^{-1})^* g_M \\ &= (\phi_i^{-1} \circ \chi_{M_i,f_i}^{-1})^*(\phi_i^* g_i - g_M) + (\phi_i^{-1} \circ \chi_{M_i,f_i}^{-1})^* g_M - (\chi_{M,f}^{-1})^* g_M \to 0 \end{aligned}$$

on Ξ with respect to the C^∞ topology. Since every point in B belongs to some domain Ξ as above if r is chosen small enough, it follows that $\bar{g}_i - \bar{g}_M \to 0$ on B with respect to the weak C^∞ topology. □

Proposition 6.3.15. *The holonomy covering of any leaf* $\operatorname{im} \hat{\iota}_{M,f}$ *of* $\widehat{\mathcal{F}}_{*,\mathrm{imm}}(n)$ *is* $\hat{\iota}_{M,f} : M \to \operatorname{im} \hat{\iota}_{M,f}$.

This proposition follows from the obvious version of Lemma 5.9.9 for $\widehat{\mathcal{M}}_{*,\mathrm{imm}}^\infty(n)$.

6.4 Universality

Definition 6.4.1. Let X be a sequential Riemannian foliated space with complete leaves, and let L_x denote the leaf through every $x \in X$, whose holonomy covering is denoted by $\widetilde{L}_x^{\mathrm{hol}}$. It is said that X is *covering-continuous* when there is a connected pointed covering $(\widetilde{L}_x, \tilde{x})$ of (L_x, x) for all $x \in X$ such that $[\widetilde{L}_{x_i}, \tilde{x}_i]$ is C^∞ convergent to $[\widetilde{L}_x, \tilde{x}]$ if $x_i \to x$ is a convergent sequence in X. When this condition is satisfied with $\widetilde{L}_x = \widetilde{L}_x^{\mathrm{hol}}$ for all $x \in X$, it is said that X is *holonomy-continuous*.

Remark 6.4.1. Observe the following:

(i) Covering-continuity and holonomy-continuity are respectively weaker than covering-determination and holonomy-determination in Definition 5.10.1, which were defined by using "if and only if" instead of "if".

(ii) The condition of being covering-continuous is hereditary (by saturated subspaces).

(iii) Covering/holonomy-continuity/determination have obvious generalizations to arbitrary Riemannian foliated spaces by using nets instead of sequences.

Example 6.4.2. The following simple examples clarify Definition 6.4.1:

(i) The Reeb foliation on S^3 with the standard metric is covering-continuous, but it is not holonomy-continuous with any Riemannian metric. If the metric is modified around the compact leaf $T^2 = S^1 \times S^1$ so that the diffeomorphism $(x, y) \mapsto (y, x)$ of T^2 is not an isometry, then this foliation becomes non-covering-continuous.

(ii) The Riemannian foliated space of [56, Example 2.5] is covering-determined but not holonomy-continuous. This example can be easily realized as a saturated subspace of a Riemannian foliated space where the holonomy coverings of the leaves are isometric to $\mathbb{R}$. So holonomy-continuity is not hereditary.

(iii) $\widehat{\mathcal{M}}^\infty_{*,\mathrm{imm}}(n)$ is holonomy-continuous. However it is not holonomy-determined for $n \geq 1$ by Remark 5.10.1 (iii), since there are different points with isometric pointed holonomy covers of the corresponding pointed leaves. To see this, take any connected complete Riemannian n-manifold M, some $x \in M$ and $f, f' \in C^\infty_{\mathrm{imm}}(M, \mathbb{E})$ such that $f(x) \neq f'(x)$. Then $\hat{\imath}_{M,f}(x) \neq \hat{\imath}_{M,f'}(x)$, but (M, x) is isometric to the holonomy covers of the pointed leaves $(\operatorname{im} \hat{\imath}_{M,f}, \hat{\imath}_{M,f}(x))$ and $(\operatorname{im} \hat{\imath}_{M,f'}, \hat{\imath}_{M,f'}(x))$.

Proposition 6.4.3 (Cf. [18, Theorem 11.4.4])**.** *For any Polish C^∞ foliated space X with complete leaves, there is a C^∞ embedding $X \to \mathbb{E}$.*

Proof. This is an adaptation of the usual argument to show the existence of C^∞ embeddings of C^∞ manifolds in Euclidean spaces [44, Theorem 1.3.4]. Let $n = \dim X$ (as foliated space), and let $B_r = B_{\mathbb{R}^n}(0,r)$ and $\overline{B}_r = \overline{B}_{\mathbb{R}^n}(0,r)$ for each $r > 0$.

Claim 6.4.1. Let Z be a Polish space, and consider the C^∞ foliated structure on $U := B_2 \times Z$ with leaves $B_2 \times \{*\}$. Let V and W be open subsets of U such that $\overline{V} \subset W$ and $\overline{W} \subset B_1 \times Z$. Then there is some $h \in C^\infty(U)$ such that $h = 1$ on V and $\operatorname{supp} h \subset W$.

Since $\overline{B}_1$ is compact, it easily follows that each $z \in Z$ has an open neighborhood P_z in Z such that, for some open subsets $G_z, H_z \subset B_2$ with $\overline{G_z} \subset H_z$ and $\overline{H_z} \subset B_1$, we have $\overline{V} \cap (B_1 \times P_z) \subset G_z \times P_z$ and $\overline{H_z} \times P_z \subset W$. Let $\{\lambda_i\}$ be a partition of unity of Z subordinated to the open cover $\{\, P_z \mid z \in Z \,\}$; in particular, for every i, there is some $z_i \in Z$ so that $\operatorname{supp} \lambda_i \subset P_{z_i}$. Let $h_i \in C^\infty(B_2)$ such that $h_i = 1$ on G_{z_i} and $\operatorname{supp} h_i \subset H_{z_i}$. Then $h_i\lambda_i \in C^\infty(U)$, $h_i\lambda_i = \lambda_i$ on $G_{z_i} \times P_{z_i}$ and $\operatorname{supp}(h_i\lambda_i) \subset H_{z_i} \times P_{z_i}$. It follows that $h = \sum_i h_i\lambda_i$ satisfies the properties stated in Claim 6.4.1.

Now, let $\mathcal{U}$ be a countable collection of C^∞ foliated charts $\phi_i : U_{2,i} \to B_2 \times Z_i$ of X such that the open sets $U_{1,i} := \phi_i^{-1}(B_1 \times Z_i)$ cover X. Using the paracompactness and regularity of X, a standard argument gives locally finite open covers, $\mathcal{V} = \{V_i\}$ and $\mathcal{W} = \{W_i\}$, with the same index set as $\mathcal{U}$, such that $\overline{V_i} \subset W_i$ and $\overline{W_i} \subset U_{1,i}$. For each i, let $\mathbb{E}_i$ be a copy of $\mathbb{E}$. Take embeddings $\psi_i : Z_i \to \mathbb{E}_i$ [28, Corollary IX.9.2]. Thus each composite

$$U_{2,i} \xrightarrow{\phi_i} B_2 \times Z_i \xrightarrow{\mathrm{id} \times \psi_i} B_2 \times \mathbb{E}_i \hookrightarrow \mathbb{R}^n \times \mathbb{E}_i =: \widetilde{\mathbb{E}}_i$$

is a C^∞ embedding with respect to the restriction of $\mathcal{F}$, which will be denoted by $\tilde{\phi}_i$. By Claim 6.4.1, there are functions $h_i \in C^\infty(U_{2,i})$ such that $h_i = 1$ on V_i and $\operatorname{supp} h_i \subset W_i$. Then a C^∞ embedding[3] $f : X \to \widehat{\bigoplus}_i \widetilde{\mathbb{E}}_i \cong \mathbb{E}$ is defined by $f(x) = \sum_a h_a(x)\tilde{\phi}_{i_{k_a}}$. □

Proof of Theorem 1.4.4. The Polish Riemannian foliated space $\widehat{\mathcal{M}}^\infty_{*,\mathrm{imm}}(n)$ has complete leaves and is holonomy-continuous (Example 6.4.2-(iii)). Thus any Polish Riemannian foliated subspace of $\widehat{\mathcal{M}}^\infty_{*,\mathrm{imm}}(n)$ is also covering-continuous (Remark 6.4.1-(ii)).

Let X be any covering-continuous Polish Riemannian foliated space with complete leaves. By Proposition 6.4.3, there is a C^∞ embedding $f : X \to \mathbb{E}$. With the notation of Definition 6.4.1, suppose that the covering-continuity of X is satisfied with the connected pointed coverings $(\widetilde{L}_x, \tilde{x}) \to (L_x, x)$ $(x \in X)$. Let $\hat{\imath}_{X,f} : X \to \widehat{\mathcal{M}}^\infty_{*,\mathrm{imm}}(n)$ be defined by $\hat{\imath}_{X,f}(x) = [\widetilde{L}_x, \tilde{f}_x, \tilde{x}]$, where $\tilde{f}_x$ is the lift of $f|_{L_x}$ to $\widetilde{L}_x$. This map is well defined because the leaves of X are complete. Moreover it is obviously foliated and continuous by the definitions of covering-continuity and the topology of $\widehat{\mathcal{M}}^\infty_{*,\mathrm{imm}}(n)$.

[3]The notation $\widehat{\bigoplus}_i \mathbb{E}_i$ is used for the Hilbert space direct sum of a family of Hilbert spaces $\mathbb{E}_i$; i.e., the Hilbert space completion of $\bigoplus_i \mathbb{E}_i$ with the scalar product $\langle (v_i), (w_i) \rangle = \sum_i \langle v_i, w_i \rangle$.

To show that $\hat\imath_{X,f}$ is C^∞, take a foliated chart $\Phi = (\chi, \Theta) : \mathcal{N}_2 \to B \times \mathcal{Z}$ of $\widehat{\mathcal{F}}^\infty_{*,\mathrm{imm}}(n)$ defined by any choice of $(V, e, \rho, \kappa, \sigma)$ as above. Let U be the domain of a foliated chart of X such that $\hat\imath_{X,f}(U) \subset \mathcal{N}_2$. Then the composite

$$U \xrightarrow{\hat\imath_{X,f}} \mathcal{N}_2 \xrightarrow{\chi} B$$

is equal to $\Pi_V \circ (f - e)$, and therefore it is C^∞.

Finally, $\hat\imath_{X,f}$ is a C^∞ embedding because the composite

$$X \xrightarrow{\hat\imath_{X,f}} \widehat{\mathcal{M}}^\infty_{*,\mathrm{imm}}(n) \xrightarrow{\mathrm{ev}} \mathbb{E}$$

equals the C^∞ embedding f. □

6.5 Realization of manifolds of bounded geometry as leaves

Proposition 6.5.1. *Let M be any connected, complete Riemannian n-manifold of bounded geometry. Then there is a C^∞ embedding $f : M \to \mathbb{E}$ such that $\widehat{\mathrm{Cl}}_\infty(\mathrm{im}\,\hat\imath_{M,f})$ is a compact subspace of $\widehat{\mathcal{M}}^\infty_{*,\mathrm{imm}}(n)$.*

Proof. Let $B_r = B_{\mathbb{R}^n}(0, r)$ for each $r > 0$. By the bounded geometry of M, there is some $r > 0$, smaller than the injectivity radius of M, such that the following properties hold:

(i) For the normal parametrizations $\kappa_x : B_r \to B_M(x, r)$ $(x \in M)$, the corresponding metric coefficients, g_{ij} and g^{ij}, as a family of C^∞ functions on B_r parametrized by x, i and j, lie in a bounded subset of the Fréchet space $C^\infty_b(B_r)$ [66, Theorem A.1], [67, Theorem 2.5] (see also [63, Proposition 2.4], [29]).

(ii) There is some countable subset $\{\, x_i \mid i \in \mathbb{N}\,\} \subset M$ and some $c \in \mathbb{N}$ such that the family of balls $B_M(x_i, r/2)$ covers M, and $B_M(x, r)$ meets at most c sets $B_M(x_i, r)$ for all $x \in M$ [71, A1.2 and A1.3], [67, Proposition 3.2].

Let $\kappa_i = \kappa_{x_i}$ for each i.

Claim 6.5.1. There is a partition of $\mathbb{N}$ into finitely many sets, $I_1, \dots, I_{c+1}$, such that $B_M(x_i, r) \cap B_M(x_j, r) = \emptyset$ for $i \in I_k$ and $j \in I_l$ with $k \neq l$.

This claim follows by considering the graph G whose set of vertices is $\mathbb{N}$, and such that there is a unique edge connecting two different vertices, i and j, if and only if $B_M(x_i, r) \cap B_M(x_j, r) \neq \emptyset$. Since there are at most c edges meeting at each vertex according to (ii),

G is $c+1$-colorable[4]; i.e., there is a partition of $\mathbb{N}$ into subsets, $I_1, \dots, I_{c+1}$, such that there is no edge joining any pair of different vertices in any I_k.

Let S be an isometric copy in $\mathbb{R}^{n+1}$ of the standard n-dimensional sphere containing the origin 0. Choose some spherically symmetric C^∞ function $\rho \in C^\infty(\mathbb{R}^n)$ such that $\rho(x) = 1$ if $|x| \le r/2$ and $\rho(x) = 0$ if $|x| \ge r$. Take also some C^∞ map $\tau\colon \mathbb{R}^n \to \mathbb{R}^{n+1}$ that restricts to a diffeomorphism $B_r \to S \setminus \{0\}$ and maps $\mathbb{R}^n \setminus B_r$ to 0. Let $\tilde\rho_i$ be the extension by zero of $\rho \circ \kappa_i^{-1}$ to the whole of M, and let $\tilde\rho^k = \sum_{i \in I^k} \tilde\rho_i$. For each k, define $f^k : M \to \mathbb{R}^{n+2}$ by

$$f^k(x) = \begin{cases} 0 & \text{if } x \notin \bigcup_{i \in I^k} B_M(x_i, r) \\ \left(\tilde\rho^k(x)/i, \tilde\rho^k(x) \cdot \tau \circ \kappa_i^{-1}(x)\right) & \text{if } x \in B_M(x_i, r) \text{ for some } i \in I^k\ . \end{cases}$$

So $f^k \circ \kappa_i = (\rho/i, \rho\cdot\tau)$, obtaining that, for every multi-index α, the function $|\partial_\alpha(f^k \circ \kappa_i)|$ is uniformly bounded over B_r by a constant depending only on $|\alpha|$. Let $f = (f^1, \dots, f^{c+1}) : M \to \mathbb{R}^{(c+1)(n+2)}$. We have $\sup_M |\nabla^m f| < \infty$ for each $m \in \mathbb{N}$ by (i). Moreover $f^k \circ \kappa_i = (1/i, \tau)$ on $B_{r/2}$, obtaining that f is a C^∞ embedding, and $\inf_M |\bigwedge^n df| > 0$ by (i). By taking any isometric linear embedding of $\mathbb{R}^{(c+1)(n+2)}$ into $\mathbb{E}$, we can consider $\mathbb{R}^{(c+1)(n+2)}$-valued functions as $\mathbb{E}$-valued functions; in particular, this applies to f.

Claim 6.5.2. $\widehat{\mathrm{Cl}}_\infty(\operatorname{im} \hat\iota_{M,f}) \subset \widehat{\mathcal{M}}^\infty_{*,\mathrm{imm}}(n)$.

This claim is true because, for all $[N, h, y] \in \widehat{\mathrm{Cl}}_\infty(\operatorname{im} \hat\iota_{M,f})$, it is easy to see that

$$\inf_N |\textstyle\bigwedge^n dh| \ge \inf_M |\bigwedge^n df| > 0\ ,$$

obtaining that h is an immersion.

Claim 6.5.3. $\widehat{\mathrm{Cl}}_\infty(\operatorname{im} \hat\iota_{M,f})$ is compact.

This assertion follows by showing that any sequence in $\operatorname{im} \hat\iota_{M,f}$ has a subsequence that is convergent in $\widehat{\mathcal{M}}^\infty_*(n)$. Assume first that the sequence is of the form $[M, f, x_{i_p}]$ for some sequence of indices i_p. Since $\mathrm{Cl}_\infty(\operatorname{im} \iota_M)$ is compact in $\mathcal{M}^\infty_*(n)$ by Theorem 5.10.3, we can suppose that $[M, x_{i_p}]$ converges to some point $[N, y]$ in $\mathcal{M}^\infty_*(n)$. Take a sequence of compact domains Ω_q in N such that $B_N(y, q+1) \subset \Omega_q$. For each q, there are pointed local embeddings $\phi_{q,p} : (N, y) \rightarrowtail (M, x_{i_p})$, for p large enough, such that $\Omega_q \subset \operatorname{dom} \phi_{q,p}$ and $\phi_{q,p}^* g_M \to g_N$ on Ω_q with respect to the C^∞ topology. Let $h_{q,p} = \phi_{q,p}^* f$ on Ω_q. It is easy to see that, for all naturals q and m, the sequence $\|h_{q,p}\|_{C^m,\Omega_q,g_N}$ is uniformly bounded. Hence the functions $h_{q,p}$ form a compact subset of $C^\infty(\Omega_q, \mathbb{R}^{(c+1)(n+2)})$ with the C^∞ topology by Proposition 5.1.11. So some subsequence $h_{q,p(q,\ell)}$ is convergent to some $h_q \in C^\infty(\Omega_q, \mathbb{R}^{(c+1)(n+2)})$ with the C^∞ topology. In fact, arguing inductively on q, it is

[4] This easily follows by induction, assigning to each i a color different from the colors of the previous vertices that are neighbors of i, which is possible because there are at most c of them (see [16]).

easy to see that we can assume that each $h_{q+1,p(q+1,\ell)}$ is a subsequence of $h_{q,p(q,\ell)}$, and therefore h_{q+1} extends h_q. Thus the functions h_q can be combined to define a function $h \in C^\infty(M, \mathbb{R}^{(c+1)(n+2)})$. Take sequences of integers, $\ell_q \uparrow \infty$ and $m_q \uparrow \infty$, so that

$$\|h - \phi^*_{q,p(q,\ell_q)} f\|_{C^{m_q},\Omega_q,g_N} = \|h_q - h_{q,p(q,\ell_q)}\|_{C^{m_q},\Omega_q,g_N} \to 0\ .$$

Then, considering h as an $\mathbb{E}$-valued function, we get that $[M, f, x_{i_{p(q,\ell_q)}}] \to [N, h, y]$ in $\widehat{\mathcal{M}}^\infty_*(n)$ as $q \to \infty$.

Now take an arbitrary sequence $[M, f, x'_p]$ in $\operatorname{im} \hat\iota_{M,f}$. By (ii), there is a sequence of naturals, i_p, such that $d_M(x'_p, x_{i_p}) < r/2$. By the above case in the proof, after taking a subsequence if necessary, we can assume that $[M, f, x_{i_p}]$ is convergent to some point $[N, h, y]$ in $\widehat{\mathcal{M}}^\infty_*(n)$. Thus, given sequences, $m_j \uparrow \infty$ in $\mathbb{N}$, and $S_j \uparrow \infty$ and $s_j \downarrow 0$ in $\mathbb{R}^+$, there is some sequence $p_j \uparrow \infty$ in $\mathbb{N}$ such that there exists some $(m_j, S_j + e^{s_j} r/2, \lambda_j, \varepsilon_j)$-pointed local quasi-equivalence $\phi_j : (N, h, y) \rightarrowtail (M, f, x_{i_{p_j}})$ for some $\lambda_j \in [1, e^{s_j})$ and $\varepsilon_j \in (0, s_j)$. Since $y'_j := \phi_j^{-1}(x'_{p_j}) \in B_N(y, e^{s_j} r/2)$, it follows that $\phi_j : (N, h, y'_j) \rightarrowtail (M, f, x'_{p_j})$ is an $(m_j, S_j, \lambda_j, \varepsilon_j)$-pointed local quasi-equivalence, showing that $[M, f, x'_{p_j}] \in \widehat{U}^{m_j}_{S_j,s_j}(N, h, y'_j)$. On the other hand, since the sequence y'_j is bounded in N, we can suppose that it is convergent to some $y' \in N$ by taking a subsequence if necessary. Hence $[N, h, y'_j] \to [N, h, y']$ in $\widehat{\mathcal{M}}^\infty_*(n)$ by the continuity of $\hat\iota_{N,h}$. Hence there are sequences, $n_j \uparrow \infty$ in $\mathbb{N}$, and $T_j \uparrow \infty$ and $t_j \downarrow \infty$ in $\mathbb{R}^+$, such that $[N, h, y'_j] \in \widehat{U}^{n_j}_{e^{s_j}T_j,t_j}(N, h, y')$ for j large enough. So

$$[M, f, x'_{p_j}] \in \widehat{U}^{m_j}_{S_j,s_j} \circ \widehat{U}^{n_j}_{e^{s_j}T_j,t_j}(N, h, y') \subset \widehat{U}^{\min\{m_j,n_j\}}_{\min\{S_j,T_j\},s_j+t_j}(N, h, y')$$

for p large enough by Proposition 6.2.2-(iv). This shows that $[M, f, x'_{p_j}] \to [N, h, y']$ in $\widehat{\mathcal{M}}^\infty_*(n)$, completing the proof of Claim 6.5.3. □

Chapter 7

Foliated spaces with trivial holonomy

This chapter contains the proofs of the results about the realization of Riemannian manifolds as leaves compact Riemannian foliated spaces stated in Sections 1.2 and 1.4.

7.1 Foliated spaces and graph colorings

This chapter will complete the proof of Theorem 1.2.1 using the results of Section 3, showing that any Riemannian manifold of bounded geometry can be realized as a leaf in a compact Riemannian foliated space without holonomy.

Let us recall the construction of X in the first sentence of Theorem 1.2.1 because it is a source of examples of compact foliated spaces with prescribed leaves. Fix a separable Hilbert space $\mathbb{E}$, and consider pairs (M, f), where $f \in C^\infty(M, \mathbb{E})$, instead of just the simply connected Riemannian n-manifold M. An isomorphism of these objects is an isometry compatible with the distinguished functions. Then, proceeding as above, equivalence classes $[M, f, x]$ can be defined by using pointed isomorphisms. They form a set $\widehat{\mathcal{M}}_*(n)$, where there is an obvious version of the C^∞ *convergence*. This convergence defines a Polish space $\widehat{\mathcal{M}}_*^\infty(n)$ [5, Theorem 1.3], whose closure operator is denoted by $\widehat{\mathrm{Cl}}_\infty$. There are also canonical maps $\hat{\imath}_{M,f} : M \to \widehat{\mathcal{M}}_*(n)$, whose images form a natural partition $\widehat{\mathcal{F}}_*(n)$. The concepts of being *non-periodic*, *locally non-periodic*, *limit-aperiodic* or *repetitive* have obvious versions for pairs (M, f) (or simply for f), obtaining $\widehat{\mathcal{M}}_{*,\mathrm{np}}^\infty(n)$ and $\widehat{\mathcal{M}}_{*,\mathrm{lnp}}^\infty(n) \equiv (\widehat{\mathcal{M}}_{*,\mathrm{lnp}}^\infty(n), \widehat{\mathcal{F}}_{*,\mathrm{lnp}}^\infty(n))$ as above, satisfying analogous properties (without requiring $n \geq 2$) [5, Section 1]; in particular, $\widehat{\mathcal{M}}_{*,\mathrm{lnp}}^\infty(n)$ is a Riemannian foliated space, whose subspace of leaves without holonomy is $\widehat{\mathcal{M}}_{*,\mathrm{np}}^\infty(n)$. This foliated space is universal among Riemannian foliated spaces satisifying a property called covering-continuity [5, Proposition 6.4]. Moreover (M, f) (or simply f) is said to be of *bounded geometry* if M is of bounded geometry and $\|\nabla^m f\|_M < \infty$ for all $m \in \mathbb{N}$. This property means that $\widehat{\mathrm{Cl}}_\infty(\operatorname{im}\hat{\imath}_{M,f})$ is compact [5, Claim 7.4]. Then Theorem 1.2.1 follows with $X = \widehat{\mathrm{Cl}}_\infty(\operatorname{im}\hat{\imath}_{M,f})$, where f is given by Proposition 1.4.5.

The construction of f in Proposition 1.4.5 will be indicated in Section 7.2. It will be reduced to Theorem 1.1.1.

7.2 Limit-aperiodic functions

Let us indicate the proof of Proposition 1.4.5 using Theorem 1.1.1. By the bounded geometry of M, there is some $0 < r < \mathrm{inj}_M$ such that the following properties hold:

(i) For the normal parametrizations $\kappa_x : B_r := B_{\mathbb{R}^n}(0,r) \to B_M(x,r)$ $(x \in M)$, the corresponding metric coefficients, g_{ij} and g^{ij}, as a family of C^∞ functions on B_r parametrized by x, i and j, lie in a bounded subset of the Fréchet space $C^\infty(B_r)$ [66, Theorem A.1], [67, Theorem 2.5] (see also [63, Proposition 2.4], [29]).

(ii) There is some countable subset $\{\, x_i \mid i \in I \,\} \subset M$ and some $c \in \mathbb{N}$ such that the balls $B_M(x_i, r/2)$ cover M, and, for all $x \in M$, $B_M(x,r)$ meets at most c balls $B_M(x_i,r)$ [71, A1.2 and A1.3], [67, Proposition 3.2]. Let $\kappa_i = \kappa_{x_i}$.

Consider the graph G with $V(G) = I$, and such that there is an edge connecting two different vertices, i and j, if and only if $B_M(x_i,r) \cap B_M(x_j,r) \neq \emptyset$. By (ii), the vertex degrees are uniformly bounded by c. So there is a coloring of G by $c+1$ colors so that adjacent vertices have different colors. This means that there is a partition of I into finitely many sets, $I_1, \dots, I_{c+1}$, such that $B_M(x_i,r) \cap B_M(x_j,r) = \emptyset$ for $i \in I_k$ and $j \in I_l$ with $k \neq l$. On the other hand, by Theorem 1.1.1, G has a limit-aperiodic vertex coloring $\alpha : I \to \{1, \dots, c\}$. Let $\alpha_i = \alpha(x_i)$.

Let S be an isometric copy in $\mathbb{R}^{n+1}$ of the standard n-dimensional sphere so that $0 \in S$. Choose some radial[1] function $\rho \in C^\infty(\mathbb{R}^n)$ such that $0 \le \rho \le 1$, $\rho(x) = 1$ if $|x| \le r/2$ and $\rho(x) = 0$ if $|x| \ge r$. Take also some C^∞ map $\tau : \mathbb{R}^n \to \mathbb{R}^{n+1}$ that restricts to a diffeomorphism $B_r \to S \smallsetminus \{0\}$ and maps $\mathbb{R}^n \smallsetminus B_r$ to 0. Let $\rho_i = \rho \circ \kappa_i^{-1}$ and $\tau_i = \tau \circ \kappa_i^{-1}$. For $k = 1, \dots, c+1$, define $f^k : M \to \mathbb{R}^{n+2}$ by

$$f^k(x) = \begin{cases} 0 & \text{if } x \notin \bigcup_{i \in I_k} B_M(x_i, r) \\ (\rho_i(x) \cdot \alpha_i, \rho_i(x) \cdot \tau_i(x)) & \text{if } x \in B_M(x_i,r) \text{ for some } i \in I_k\,. \end{cases}$$

Fix a linear injection $\mathbb{R}^{(c+1)(n+2)} \subset \mathbb{E}$. Then

$$f = (f^1, \dots, f^{c+1}) : M \to \mathbb{R}^{(c+1)(n+2)} \subset \mathbb{E}$$

[1] A function of the radius in polar coordinates.

is a C^∞ immersion, and therefore it is locally non-periodic. Moreover f is of bounded geometry and limit-aperiodic, as follows from (i), and from the bounded geometry and limit-aperiodicity of α.

If M is repetitive, then this property can be easily used to choose the points x_i so that the pair $(M, \{x_i\})$ is *repetitive* in an obvious sense (as a Riemannian manifold with a distinguished subset). With this condition, G is repetitive, and α can be assumed to be repetitive by Theorem 1.1.1. It follows that f is also repetitive, showing Proposition 1.4.5.

Smaller subspaces, $\widehat{\mathcal{M}}^\infty_{*,\mathrm{imm}}(n) \subset \widehat{\mathcal{M}}^\infty_{*,\mathrm{lnp}}(n)$ and $\widehat{\mathcal{M}}^\infty_{*,\mathrm{emb}}(n) \subset \widehat{\mathcal{M}}^\infty_{*,\mathrm{np}}(n)$, are defined by requiring the distinguished functions to be C^∞ immersions or C^∞ embeddings. It turns out that $\widehat{\mathcal{M}}^\infty_{*,\mathrm{imm}}(n)$ is Polish and dense in $\widehat{\mathcal{M}}^\infty_{*}(n)$ [5, Theorem 1.4]. Thus we get a C^∞ and Riemannian foliated subspace, $\widehat{\mathcal{M}}_{*,\mathrm{imm}}(n) \equiv (\widehat{\mathcal{M}}^\infty_{*,\mathrm{imm}}(n), \widehat{\mathcal{F}}^\infty_{*,\mathrm{imm}}(n))$, where $\widehat{\mathcal{M}}^\infty_{*,\mathrm{emb}}(n)$ is a union of leaves without holonomy. In fact, we can use the distinguished immersions to define its foliated charts more easily [5, Theorem 1.4]. Then there is some $h \in C^\infty(M, \mathbb{E})$ such that $\widehat{\mathrm{Cl}}_\infty(\operatorname{im} \hat{\iota}_{M,h})$ is a (minimal) compact subspace of $\widehat{\mathcal{M}}^\infty_{*,\mathrm{emb}}(n)$. This slight sharpening of Proposition 1.4.5 can be easily proved as follows. Let $f \in C^\infty(M, \mathbb{E})$ be given by Proposition 1.4.5, inducing the foliated space $X = \widehat{\mathrm{Cl}}_\infty(\operatorname{im} \hat{\iota}_{M,f})$. Then there is a C^∞ embedding $\tilde{h} : X \to \mathbb{E}$ [18, Theorem 11.4.4], and the function $h = \hat{\iota}^*_{M,f}\tilde{h} \in C^\infty(M, \mathbb{E})$ satisfies the above property. However $\widehat{\mathrm{Cl}}_\infty(\operatorname{im} \hat{\iota}_{M,h}) \equiv X$ (no new foliated space is produced with this sharpening).

Distinguished subsets of Riemannian manifolds can be used instead of distinguished functions to construct a Riemannian foliated space, producing also compact Riemannian foliated spaces with a prescribed leaf [13].

7.3 Realization of Riemannian coverings of compact manifolds

Let (M, g) be a Riemannian manifold, and let G be a quotient group of $\pi_1(M)$. Let (M', g') be the regular Riemannian covering space induced by G, and let $\pi \colon M' \to M$ denote the quotient map. Let $p \in M$. Then G acts freely on (M', g') by deck transformations, and Gp is a separated net in X. Choose a finite symmetric generating set S of G, determining a Cayley graph structure on G. Using the same arguments as in Chapter 7, we get that to each limit-aperiodic (repetitive) coloring of G we can associate a (minimal) Riemannian foliated space without holonomy with a leaf isometric to M'.

This method can be used to provide many examples of compact Riemannian foliated spaces realizing a prescribed leaf. For example, let M' be the Riemannian universal covering space of some compact Riemannian manifold M with $\pi_1(M) = \mathbb{Z}$. Then any limit-aperiodic (repetitive) coloring of $\mathbb{Z}$ determines a compact Riemannian (minimal) foliated space with a leaf isometric to M'. It is easy to construct such colorings. E.g., one can take a coloring of $\mathbb{Z}$ such that over $\mathbb{Z}_{\geq 0}$ and $\mathbb{Z}_{<0}$ coincides with the Thue-Morse sequence using different colors.

Part III

Versión en galego

Chapter 8

Resumo en galego

Esta tese está composta por dúas partes principais. A primeira está adicada a probar que, para todo grafo X infinito, conexo, (repetitivo) e con valencia nos vértices uniformemente limitada por $\Delta < \infty$, hai unha coloración (repetitiva) límite-aperiódica usando Δ cores. Así mesmo, deste teorema derívanse varias consecuencias directas, por exemplo a existencia de coloracións límite-aperiódicas (repetitivas) para toda teselación (repetitiva) dunha variedade de Riemann. Na segunda parte demóstrase que toda variedade de Riemann (repetitiva) de xeometría limitada se pode realizar isométricamente como folla dun espazo foliado de Riemann compacto (e minimal), cuxas follas teñen holonomía trivial. Para chegar a este teorema úsase o resultado previo sobre coloracións, ademáis dunha cantidade considerable de resultados técnicos acerca do espazo de variedades de Riemann punteadas coa topoloxía definida pola converxencia C^∞. As seguintes seccións conteñen descricións máis precisas dos obxectivos e resultados da tese.

8.1 Coloracións de grafos

Sexa (X, E) (ou simplemente X) un grafo (non dirixido) simple[1] e conexo. Dado un subconxunto F dos números naturais, unha coloración $\phi\colon X \to F$ é *non-periódica*, *aperiódica*[2], ou *diferenciadora* se non hai automorfismos non triviais de X que conserven ϕ. O *número diferenciador*, denotado por $D(X)$, é o enteiro positivo máis pequeno tal que existe algunha coloración non-periódica ϕ de X usando $D(X)$ cores. Albertson and Collins introduciron este concepto en [2], e o cálculo de $D(X)$ (ou ben o cálculo de límites superiores) para diversas familias de grafos foi obxecto de investigación nos últimos anos (véxase [47], [26]).

[1] Un grafo é *simple* se cada par de vértices está contido en como moito unha aresta.

[2] Nalgunhas publicacións, este termo usouse co significado que nesta tese designamos como límite-aperiódico.

Outro grafo simple e conexo Y é *límite* de X se, usando a distancia natural do grafo d_Y, para cada $n \in \mathbb{N}$ e $y \in Y$, podemos atopar unha copia isomorfa da bola $B_Y(y,n)$ de Y en X. Analogamente, pódese definir cando un grafo coloreado (Y,ψ) é o límite de (X,ϕ). A coloración $\phi\colon X \to F$ é *límite-aperiódica* ou *límite-diferenciadora* se cada grafo coloreado límite (Y,ψ) é diferenciador, e o *número diferenciador no límite*, denotado por $D_L(X)$, é o menor $n \in \mathbb{N}$ tal que existe unha coloración límite-diferenciadora usando n cores.

Un grafo X (respectivamente, un grafo coloreado (X,ϕ)) é *repetitivo* se cada padrón de X (respectivamente, de (X,ϕ)) aparece uniformemente en X con respecto á distancia natural do grafo d_X. Sexa X un grafo con valencia máxima $\deg X < \infty$. O primeiro resultado principal desta tese afirma que $D_L(X) \leq \deg X$ para todo grafo infinito que satisfaga os requisitos anteriormente enunciados.

Se X é un grafo finito, entón a aperiodicidade no límite é equivalente á aperiodicidade, e polo tanto temos $D_L(X) = D(X)$. Neste caso, probouse en [48] que $D(X) \leq \deg X$ excepto para as seguintes excepcións, nas que $D(X) \leq \deg X + 1$: o grafo completo K_n con n vértices ($n \geq 2$), o grafo (n,n)-bipartito $K_{n,n}$ ($n \geq 1$), e o grafo cíclico C_5 de 5 vértices. Por esta razón, nesta tese trataremos só grafos infinitos.

Teorema 8.1.1. *Sexa X un grafo infinito conexo e simple tal que $\Delta := \deg X < \infty$. Entón existe unha coloración límite-aperiódica ϕ de X usando Δ cores. Se, ademáis, X é repetitivo, entón existe unha coloración límite-aperiódica e repetitiva de X usando Δ cores.*

Sexa X un grafo simple e conexo. Unha coloración do conxunto de arestas, $\phi : E(X) \to F$, é unha *arco-coloración* de X, e (X,ϕ) é un *grafo arco-coloreado*. Os conceptos de *isomorfismo* (*punteado*), grafos arco-coloreados (puntuadamente) *isomorfos*, e grupos de *automorfismos* de grafos arco-coloreados (puntuados) son xeneralizacións directas das nocións correspondentes a grafos coloreados (nos vértices) (puntuados). Entón podemos definir as nocións de grafos arco-coloreados *aperiódicos*, *límite-aperiódicos* e *repetitivos* do mesmo xeito co que as definimos para grafos coloreados (nos vértices) na Sección 2.3.

Supoñamos agora que X é infinito e ten valencia máxima limitada $\deg X < \infty$. Entón o *grafo dual* ou *grafo de lias* asociado, denotado por $L(X)$, se pode definir do seguinte xeito: cada vértice en $L(X)$ representa unha aresta en X, e dous vértices en $L(X)$ son adxacentes se e só se as arestas correspondentes en X comparten un vértice. $L(X)$ é un grafo simple satisfacendo $\deg L(X) < 2(\deg X - 1)$. Da definición de $L(X)$ séguese que existe unha correspondencia obvia entre coloracións de vértices de $L(X)$ e coloracións de arestas de X. Polo Teorema forte de isomorfismo de Whitney [76], temos que X e $L(X)$ determínanse unívocamente un ao outro salvo isomorfismo, e ademáis hai unha correspondencia biunívoca entre isomorfismos de X e de $L(X)$. A construción de $L(X)$ a partir de X é de natureza local, no senso de que, dada unha aresta e de X cuxo vértice correspondente é $v_e \in L(X)$, a clase de isomorfía punteada de $(B_{L(X)}(v_e, r), v_e)$ está determinada pola clase de isomorfía punteada de $(B_X(u, r+1), u)$, onde u é un vertice adxacente a e. Este feito, xunto co Teorema forte de isomorfismo de Whitey, implican que, se $L(Y)$ é un límite de $L(X)$, entón o grafo Y é un límite de X. Entón podemos deducir o seguinte resultado.

Teorema 8.1.2. *Sexa X un grafo simple* (*repetitivo*) *e infinito de valencia máxima finita* $\deg X < \infty$. *Entón X admite unha arco-coloración límite aperiódica* (*e repetitiva*) *usando* $\deg L(X) \leq 2(\deg X - 1)$ *cores.*

A continuación presentaremos unha exposición informal que ilustra como se pode demostrar este resultado. Sexa ϕ unha coloración límite-aperiódica de $L(X)$. Entón temos unha arco-coloración $\hat{\phi}$ en X inducida por ϕ. Sexa $(Y, \hat{\psi})$ un grafo arco-coloreado que é un límite de $(X, \hat{\phi})$, e supoñamos que $(Y, \hat{\psi})$ ten algún isomorphismo non trivial $h \neq \mathrm{id}$. Entón, segundo o exposto anteriormente, $\hat{\psi}$ induce unha coloración nos vértices de $(L(Y), \psi)$, e h induce unha isometría non trivial $\hat{h}$ de $(L(Y), \psi)$. Pero $(L(Y), \psi)$ ten que ser un límite de $(L(X), \phi)$, o cal contradí a hipótese de que ϕ é límite-aperiódica. A demostración de que ϕ pode ser repetitiva procede de xeito similar.

O análogo do número diferenciador cando se consideran arco-coloracións en lugar de coloracións de vértices chámase o *índice diferenciador*, denotado por $D'(X)$. O valor de $D'(X)$ para diversas familias de grafos estudouse en [3], [4]. O límite superior $D'(X) \leq \deg X$ foi demostrado para X infinito en [15]. Pódese definir o *índice diferenciador no límite* dun grafo simple e conexo X, denotado por $D'_L(X)$, como a menor cantidade de cores necesaria para producir unha arco-coloración de X límite-aperiódica. Entón o Teorema 1.1.2 produce un límite superior para o índice diferenciador no límite.

De feito, é posible que unha demostración similar á presentada no Capítulo 3 poda producir unha arco-coloración límite-aperiódica usando $\deg X$ cores para todo grafo simple e infinito X que satisfaga $\deg X < \infty$. Para isto, escollemos un punto $p \in X$. Podemos construír arco-coloracións aperiódicas usando $\deg X$ cores utilizando as mesmas ideas da Proposición 3.6.27. Máis concretamente, usamos a cor 0 para diferenciar o punto p e o resto de cores utilizando o análogo de BFS-orderings para arestas. Entón, de xeito similar a como se fai no Capítulo 3, quizáis se pode demostrar o seguinte resultado.

Conxectura 8.1.3. Sexa X un grafo simple e infinito de valencia máxima limitada $\deg X < \infty$. Entón existe unha arco-coloración límite aperiódica usando $\deg X$ cores. Se ademáis X é un grafo repetitivo, entón a coloración tamén pode ser repetitiva.

Unha aplicación interesando do Teorema 8.1.1 é a proba da existencia de teselacións coloreadas límite-aperiódicas. Por simplicidade, consideremos unha teselación T dunha variedade de Riemann de dimension n con esquinas, M, usando teselas que se unen en caras. Estas teselas deben ser isométricas a un conxunto finite de prototeselas, $\mathcal{T}$, que cosisten en variedades de Riemann compactas con bordo e de dimensión n (véxase [14] para a definición de teselacións en espazos máis xerais). As isometrías teseladas de (M, T) son as isometrías de M que levan teselas en teselas. Usando ditas isometrías teseladas, podemos definir as nocións de aperiodicidade no límite e repetitividade neste contexto. Do mesmo xeito, as teselacións coloreadas e teselacións coloreadas nas caras teñen un sentido obvio, así como a aperiodicidade no límite e a repetitividade para estes tipos de teselacións.

Pódese asociar a T de xeito canónico un grafo X, cun vértice v_t asociado a cada tesela $t \in T$, e declarando que v_t e adxacente a $v_{t'}$ se e só se t e t' están unidas nunha cara de dimensión $n-1$. Entón, se M non é compacta, X é un grafo infinito de valencia máxima finita, e todo isomorfismo teselado de T induce un isomorfismo de X. Sexa $\phi\colon X \to [\deg X] := \{0, 1, \ldots, \deg X - 1\}$ unha coloración límite-aperiódica de X. Esta coloración induce unha teselación coloreada límite-aperiódica T' usando teselas coloreadas isométricas ao conxunto finito de prototeselas coloreadas $\mathcal{T}' := \mathcal{T} \times [\deg X]$. Ademáis, se a teselación é repetitiva, entón a teselación coloreada resultante pódese construir repetitiva. En resumo, temos o seguinte resultado.

Teorema 8.1.4. *Sexa T unha teselación usando unha cantidade finita de prototeselas que están unidas por caras, e sexa Δ o máximo número de caras $(n-1)$-dimensionais das prototeselas. Entón existe unha coloración límite-aperiódica da teselación usando Δ cores. Se T é repetitiva, entón a coloración tamén se pode construir repetitiva.*

Coloreando as caras das teselas en lugar das propias teselas, podemos deducir o seguinte resultado do Teorema 8.1.2.

Teorema 8.1.5. *Sexa T unha teselación usando unha cantidade finita de prototeselas que se unen en caras, e sexa Δ o número máximo de caras $(n-1)$-dimensionais das prototeselas. Entón existe unha coloración das caras límite-aperiódica usando $2(\Delta-1)$ colors. Se T é repetitivo, entón a coloración tamén se pode construir repetitiva.*

O Teorema 8.1.1 tamén se usará para realizar variedades de Riemann de xeometría limitada como follas de espazos foliados de Riemann compactos. Pódese atopar unha explicación máis detallada na Sección 8.2.

Para obter unha idea intuitiva da demostración do Teorema 1.1.1, consideremos o caso no que queremos romper simetrías non triviais en grafos finitos. Supoñemos entón que X é finito, con valencia máxima $\deg X$. A continuación detallamos un procedemento sinxelo para construir unha coloracíon aperiódica. Escollemos un punto $x \in X$, ao cal lle asignamos a cor 0. Agora, se x é o único punto coa cor 0, entón calquera automorfismo de grafos $h\colon X \to X$ debe fixar o punto x. A esfera $S(x,1)$ ten como moito $\deg X - 1$ puntos, polo que se coloreamos a esfera de tal xeito que non hai dos puntos nela coa mesma cor, entón h debe fixar todos os puntos de $S(x,1)$. Este procedemento pode continuar por indución usando unha relación de orde en X, e ao rematar obtemos unha coloración aperiódica usando $\deg X$ cores. Ademáis, se X é tal que existen puntos x, y a distancia suficientemente grande, entón é fácilmente comprobable que se pode reusar a cor 0 para obter motas coloracións aperiódicas diferentes usando este método.

Tendo esta construción en mente, a demostración procede máis ou menos do xeito seguinte. En primeiro lugar, dividimos o grafo $X \equiv X_{-1}$ en parches finitos e conexos de tamaño limitado, de tal xeito que os seus centros formen un conxunto de Delone $X_0 \subset X_{-1}$. Entón, para cada parche, construímos unha cantidade suficientemente grande de coloracións diferentes $\phi^i_{-1,x}$. Se escollemos unha coloración $\phi^i_{-1,x}$ para cada $x \in X_0$, podemos pensar que define unha coloración en X_0 que leva x a i. Pódese dotar o conxunto X_0 cunha estrutura de grafo de tal xeito que, se a coloración inducida en X_0 leva puntos próximos a cores diferentes, entón obtemos o seguinte resultado parcial para a combinación das coloracións $\phi^i_{-1,x}$ e $R, S > 0$: se hai un isomorfismo de grafos preservando a coloración entre $(B_X(x,R), x)$ e $(B_X(y,R), y)$, entón ou ben $x = y$ ou $d(x,y) > S$.

A condición de aperiodicidade no límite é precisamente unha familia numerable de condicións de este tipo. Deste xeito, xeneralizamos a discusión precedente dividindo X_0 en parches, definindo un grafo $X_1 \subset X_0$ tal que as coloracións nos parches definen unha coloración X_1, etc. Usando un argumento diagonal, obtemos a coloración desexada.

8.2 Realización de variedades como follas

Recordemos que un espazo foliado $X \equiv (X, \mathcal{F})$ de dimensión n é un espazo topolóxico X dotado dunha partición $\mathcal{F}$ en variedades conexas (follas) de tal xeito que X se pode describir localmente como un produto $B \times Z$, onde B é unha bola aberta en $\mathbb{R}^n$ e Z é un espazo topolóxico arbitrario (transversal local), e os subconxuntos $B \times \{*\}$ corresponden a abertos nas follas. $\mathcal{F}$ é unha estutura foliada ou laminación. Comumente se asume que os espazos foliados son polacos[3] para ter mellores propiedades. Moitas nocións sobre foliacións se poden extender a espazos foliados, por exemplo cartas foliadas, placas, atlas foliados, pseudogrupo de holonomía, grupo de holonomía e cobertura de holonomía das follas, minimalidade, transitividade, funcións foliadas, etc. Algúns resultados básicos

[3]Recordemos que un espazo topolóxico é *polaco* se é separable e completamente metrizable

tamén se poden estender; por exemplo, hai unha versión obvia to teorema de estabilidade local de Reeb, e a unión das follas con holonomía é un subespazo magro se X é segundo numerable. Varias clases interesantes de espazos foliados aparecen de xeito natural en diversas areas das matemáticas, como na dinámica, aritmética, teselacións, grafos e foliacións (subconxuntos minimais).

Unha estutura foliada C^∞ ven dada por un atlas foliado cuxos cambios de coordenadas son C^∞ na dirección das follas, con derivadas na dirección das follas de orde arbitraria continuas no espazo ambiente. Isto da lugar ao concepto de espazo foliado C^∞. Para salientar a diferenza, referirémonos á estrutura foliada subxacente unha estrutura foliada C^∞ como estutura foliada topolóxica. Nun espazo foliado C^∞ $X \equiv (X, \mathcal{F})$, o conecpto de función C^∞ defínese pedindo que as súas expresións locais, usando coordenadas foliadas, son C^∞ na dirección das follas, con derivadas na dirección das follas de orde arbitraria continuas no espazo ambiente. Fibrados e seccións C^∞ tamén teñen sentido en X, definidos pedindo que as descripcións locais veñan dadas por funcións C^∞ no senso anterior. Por exemplo, o fibrado tanxente TX (ou $T\mathcal{F}$) é o fibrado vectorial C^∞ en X que consiste en vectores tanxentes ás follas, e unha métrica de Riemann en X consiste en métricas de Riemann nas follas que xuntas forman unha sección C^∞ de X. Isto da lugar ao concepto de espazo foliado de Riemann.

En particular, se X é unha variedade, entón $(X, \mathcal{F})$ é unha variedade foliada, e $\mathcal{F}$ é unha foliación.

O segundo problema tratado nesta tese é a realización de varieades de Riemann como follas de espazos foliados de Riemann compactos. Esta é unha variación do problema de realizar variedades como follas de variedades foliadas compactas, o cal ten unha longa historia con contribucións por parte de matemáticos de primer nivel.

A teoría de foliacións, como area de investigación, foi iniciada por Reeb, Ehreshman e Haefliger. Reeb construiu a primeira foliación de S^3 usando o que agora se denomina compoñente de Reeb. Máis adiante, Novikov probou que unha foliación de S^3 debe conter polo menos unha compoñente de Reeb. Estas ideas, así como a estrutura topolóxica dos fluxos (teoría de Poincaré-Bendixon, o fluxo de Denjoy e o fluxo de Cherry do toro), asentáronse nunha posición relevante na investigación matemática no inicio da década dos setenta, obtendo a atención de matemáticos da categoría de Hirsch, Thurston, Plante, Mossu, Pelletier, Anosov, Ruelle-Sullivan, Raymond, Ghys, Inaba, Duminy, etc. En particular, sendo unha pregunta que aparece de xeito natural, Sullivan [74] e Sondow [72] preguntáronse que variedades poden ser realizadas como follas de variedades compactas. Con posterioridade, probouse que toda superficie se pode realizar como folla dunha foliación de codimensión un unha variedade compacta [20]. Non obstante, isto non é certo para variedades de dimensión superior [33], [49], [11], [73], [70].

Unha folla dunha variedad foliada compacta e diferenciable $(M, \mathcal{F})$ posúe unha clase canónica de case-isometría de métricas de Riemann, representada pola restrición de calquera métrica de Riemann on M. Entón, como unha versión métrica deste problema, tamén era natural preguntarse que tipos de case-isometría se poden realizar como follas de variedades foliadas compactas. De feito este problema ten unha extensión obvia a espazos foliados compactos C^∞. Unha publicación interesante tratando esta estrutura métrica das follas foi escrita por D. Cass [21], quen deu os primeiros resultados publicados relacionando as propiedades de recurrencia da follas de foliacións cos seus tipos de case-isometría, e citou un resultado non publicado de Gromov, que foi desenvolvido máis adiante en [6].

Hai exemplos de variedades de Riemann conexas de xeometría limitada cuxos tipos de case-isometría non se poden realizar como follas de foliacións de codimensión un en variedades pechadas [11], [78], [68], [69]. Véxase [46] para ter unha lectura histórica desdes desenvolvementos.

A xeometría limitada xoga un papel importante nestes resultados. Recordemos que unha variedade de Riemann M é de xeometría limitada cando ten un radio de inxectividade positiva, e a m-ésima derivada covariante do tensor de curvatura ten a norma uniformemente limitada para toda orde m; en particular, M é completa pola positividade do radio de inxectividade. A continuación presentamos exemplos de variedades de xeometría limitada: coberturas de variedades de Riemann conexas e pechadas, grupos de Lie conexos con métricas invariantes pola esquerda, e follas de espazos foliados de Riemann compactos. Pódense construir máis exemplos usando perturbacións con soporte compacto en variedades de Riemann de xeometría limitada. De feito, toda variedade C^∞ admite unha métrica de xeometría limitada [35]. En contra do que as anteriormente mencionadas construcións de "non follas" de xeometría limitada en codimensión un podería levar a pensar, probamos o seguinte teorema, que é o segundo resultado principal desta teses.

Teorema 8.2.1. *Toda variedade de Riemann conexa de xeometría limitada é isométrica a unha folla dun espazo foliado de Riemann compacto e sen holonomía. Ademáis, se a variedade é repetitiva, entón o espazo foliado se pode construir minimal.*

Polo tanto, o estudo xeral das follas de espazos foliados de Riemann compactos (e minimais) é o estudo das variedades de Riemann de xeometría limitada (e repetitivas). Dado que toda variedade C^∞ admite métricas de Riemann de xeometría limitada [35], obtemos o seguinte corolario.

Corolario 8.2.2. *Toda variedade diferenciable M se pode realizar como folla dun espazo foliado de Riemann compacto e sen holonomía X.*

É unha opinión común que polo menos a primeira parte do Teorema 1.2.1 debería ser certa, e que se pode demostrar usando a clausura do mergullamento canónico da variedade no espazo de Gromov $\mathcal{M}_*$ de espazos métricos propios e punteados [36], [37, Chapter 3], ou, aínda mellor, na versión diferenciable, o espazo $\mathcal{M}_*^\infty(n)$ de clases de isometría punteada de variedades de Riemann conexas e completas de dimensión n, coa topoloxía definida pola converxencia C^∞ ([60, Chapter 10, Section 3.2] e Theorem 1.3.2). Non obstante, o autor non coñece ningunha demostración publicada antes de [5] e [6].

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